Chapter 8: Graphical analysis *Section 2*

First derivative analysis

What you need to know already:	*What you can learn here:*
➤ How to use derivatives to identify the critical values of a function and its extreme points. ➤ Fermat's theorem on critical values.	➤ How to use algebraic methods, without a calculator, to identify the entire pattern of increase/decrease of a function and to classify all cut points of the derivative.

Fermat's theorem gives us a method for identifying extreme points, but more information is needed to determine the pattern of the graph *between* critical values. Identifying such pattern can also allow us to classify the critical values, and any other special values, without the use of a calculator.

Fortunately, identifying this pattern is very simple, since we only need to remember the graphical meaning of the derivative.

Knot on your finger

Given a differentiable function $f(x)$:

- At any point $(c, f(c))$ where $f'(c) > 0$, the slope of the graph is positive and hence the function is ***increasing***.
- At any point $(c, f(c))$ where $f'(c) < 0$, the slope of the graph is negative and hence the function is ***decreasing***.

Now, if we think about it, there are only four options for what can happen to the derivative at a value *c*:

1) $f'(c) = 0$, corresponding to a critical number
2) $f'(c) = DNE$, corresponding to a cut point that may be a critical number
3) $f'(c) > 0$, corresponding to an interval of increase
4) $f'(c) < 0$, corresponding to an interval of decrease

This small number of options allows us to develop a simple method to analyze the up-down pattern of the graph of a function. This strategy is based on the strategy for solving an inequality, since we are really trying to solve/analyze the inequality:

$$f'(x) > 0$$

by using its cut points. Therefore, you may want to review that strategy, in case you are rusty on it.

Strategy for performing a First derivative analysis

To determine the ***pattern of increase/decrease*** in the graph of a function $y = f(x)$ and to identify and classify the cut points of $f'(x)$:

1. Use standard algebraic methods to ***find the cut points*** of $f'(x)$.
2. ***Place*** these values on a ***number line***, as in the strategy to solve an inequality.
3. ***Test*** each resulting interval, to see if the function is increasing or decreasing there, and indicate this information on the number line.
4. ***Follow the*** increase/decrease ***pattern*** and use any other feature of the function to ***classify*** each critical number as a maximum, minimum or other feature.

The last step is also known as the ***First Derivative Test***.

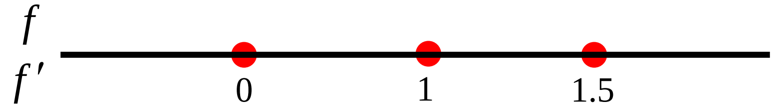
Example: $f(x) = \dfrac{x^3}{x-1}$

Step 1: we first compute the derivative:

$$f'(x) = \frac{3x^2(x-1) - x^3}{(x-1)^2} = \frac{2x^3 - 3x^2}{(x-1)^2} = \frac{x^2(2x-3)}{(x-1)^2}$$

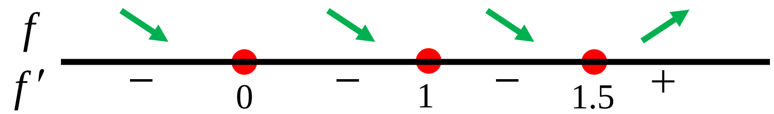
Then we identify the cut points. These occur when the numerator is 0, which is at $x = 0$ and $x = 1.5$, and when the denominator is 0, that is, at $x = 1$. Notice that the last cut point is not a critical number, since it is not in the domain of the original function.

Step 2: We place all the cut points on a number line:



Notice that when placing the cut points on the number line it is important to keep their order, but there is no need to keep them in scale.

Step 3: We test the derivative in each interval to determine if it is positive (increasing) or negative (decreasing). Remember that all we need to know is whether the derivative is positive or negative. Since our derivative includes two squares, which are always positive, we only need to focus on the factor $(2x-3)$. This is negative before 1.5 and positive after that. Therefore, the pattern is as follows:

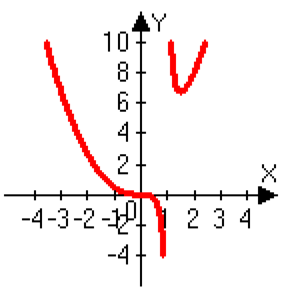


Step 4: What do we have at the three critical values?

At $x = 0$ the function is differentiable, but the function is always decreasing, so it is neither a max, nor a min.

At $x = 1$ the original function is undefined and of the form #/0, so we have a vertical asymptote.

At $x = 1.5$ the function is differentiable and it changes from going down to going up. Therefore we have a minimum at $(1.5,\ f(1.5))$.



We did everything without using the calculator, however, we can still rely on it to check that our conclusions are reasonable. The graph we obtain in this way confirms that.

Example: $y = x\ln x^2$

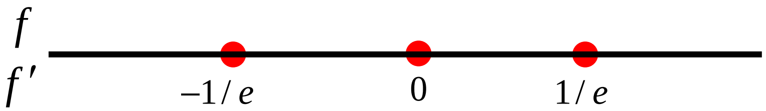
If you feel tempted to rewrite the function as $y = 2x\ln x$, think again: why is this NOT the same function?

1) The derivative is $y = \ln x^2 + 2$, so that its cut points occur when:

$$\ln x^2 + 2 = 0 \quad \Rightarrow \quad \ln x^2 = -2 \quad \Rightarrow \quad x^2 = e^{-2}$$

Therefore we need to look at $x = \pm\sqrt{e^{-2}} = \pm\frac{1}{e}$. But wait! There is another option to consider, namely, when the derivative is undefined. This occurs at $x = 0$, where the logarithm is undefined.

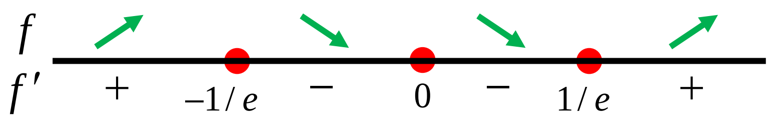
2) We place all these values on a number line:



3) We test each interval, but in this case we do not even need to pick specific numbers, as it is sufficient to consider their relative positions.

We notice that in the first and last interval the value of x^2 is large, so its logarithm is positive and so is the derivative.

In the two middle intervals the value of *x* is close to 0, so that x^2 is very small and its logarithm is negative enough to make the whole derivative negative. Therefore the pattern is as follows:



4) To classify the cut points, we notice that at $x = \pm 1/e$ the function is differentiable and the pattern tells us that we have a maximum at $x = -1/e$ and a minimum at $x = 1/e$. What do we get at $x = 0$? At this point we cannot determine that by hand, as we need to resolve the limit:

$$\lim_{x \to 0} x \ln x^2$$

This limit requires a method called L'Hospital's rule, which we have not seen yet. There is always more to learn!

Please notice that you may be used, from your high school days, to perform only the ***First Derivative Test***. This, as I mentioned, consists of only step 4 of the first derivative analysis and its purpose is only to classify critical points. Although you are expected to know what this test is and does, you should always perform a first derivative ***analysis***, with all its four steps, and not just the test portion.

Summary

- We analyze the first derivative by computing it, analyzing it as in the method for solving inequalities and then drawing conclusions based on the pattern we see on the number line graph.

Common errors to avoid

- When asked to analyze the first derivative, perform all four steps, not just the *first derivative test* step that comes at the end.

Learning questions for Section D 8-2

Review questions:

1. Describe how to perform a first derivative analysis.
2. Describe how to classify the cut points of the first derivative by using the information about the rest of the pattern.
3. Explain the difference between a first derivative test and a first derivative analysis.

Memory questions:

1. How is the derivative of a function at a point where the function is increasing?
2. How is the derivative of a function at a point where the function is decreasing?
3. What graphical feature occurs at a point where the function is continuous and its derivative changes from positive to negative?
4. What graphical feature occurs at a point where the function is continuous and its derivative changes from negative to positive?
5. What is the purpose of the first derivative analysis?
6. What is the purpose of the first derivative test?

Computation questions:

Perform a full first derivative analysis on the each of the functions presented in questions 1-41.

1. $y = x^5 - 10x^3$
2. $y = x^4 - 4x^2$
3. $f(x) = \dfrac{x}{x^2 - 4}$.
4. $y = \dfrac{x^2}{1 + x}$
5. $y = \dfrac{x + 1}{x^2 - x}$
6. $y = \dfrac{1}{x^3 - x^2}$
7. $y = \dfrac{1}{x^3 - x}$
8. $f(x) = \dfrac{x + 1}{x^2 - 4}$.

9. $y=\frac{x^3+1}{x^2+1}$. Use the fact the one of the critical values is at $x\approx 0.6$.

10. $f(x)=\frac{x^3+2}{x^3-8}$

11. $f(x)=\sqrt{3x^4+2x^2}$.

12. $y=x-\sqrt[3]{x},\ -4\le x\le 4$

13. $f(x)=\frac{\sqrt{x}}{x^2+4}$

14. $y=(x-1)^{2/3}-(x+1)^{2/3}$

15. $y=3x^{1/3}+2x^{-2/3}$

16. $y=\left|x^3-x\right|$. (Remember that $|x|=\sqrt{x^2}$)

17. $f(x)=\sqrt{x^2-x^3}$

18. $f(x)=\sqrt{\left(x^3-x\right)^2}$.

19. $f(x)=\frac{2x}{\sqrt{x^2-5}}$

20. $f(x)=\frac{x-1}{\sqrt{x^2+2}}$

21. $y=\left(9-x^2\right)^{1/3}$

22. $y=\left(1-x^3\right)^{1/5}$

23. $y=e^x-e^{2x}$ on [-1, 1]

24. $y=x^2e^x$

25. $y=e^x\left(e-e^x\right)$

26. $y=e^{x^2-x^3}$

27. $f(x)=\frac{2e^x}{x^2}$

28. $f(x)=\frac{2x^2}{e^x}$

29. $y=2e^{-x^2}-e^{-4x^2}$

30. $f(x)=\frac{e^{x^2}}{x}$

31. $y=\frac{e^{x^2}}{1+e^{x^2}}$.

32. $y=\frac{e^x}{1+e^x}$

33. $f(x)=\sinh x^2-2x^2$.
You can assume that $\cosh(1.3)=2$.

34. $f(x)=\cosh x^2-x^2$

35. $y=\ln\left(x^2+2x+2\right)$

36. $y=\ln\left(x^2-x^3\right)$

37. $y=\frac{x}{\ln x}$ on [1, 4].

38. $y=\frac{x}{\ln x}$

39. $y=x+\tan x, 0\le x\le\pi$.

40. $y=x+2\sin x,\quad -3\le x\le 3$

41. $y=\tan x^2$ for $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$

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42. Perform a full first derivative analysis on the parametric curve $x=\cos t,\ y=\sin^3 t$ and use it to determine the coordinates of the highest and lowest points on this curve.

43. Perform a full first derivative analysis on the parametric curve $\left(\cos t, e^{\sin t}\right)$ and use it to determine the coordinates of the highest and lowest points on this curve.

Theory questions:

1. If the derivative of a function $f(x)$ changes at $x = c$ from positive to negative, which two possible graphical features may be occurring there?

2. Can we know whether an extreme value is absolute or local by using only the first derivative?

3. How do we identify the presence of a maximum from the first derivative analysis?

Application questions:

1. The manager of a company that produces a certain type of gadgets notices that the cost per item of producing *n* gadgets in a month is given by the function $C = 250 + 10.7n - 0.3n^3$. Determine the values of *n* for which C' is positive and those for which it is negative and explain what these values represent in the problem.

Templated questions:

1. Perform a first derivative analysis on each of the functions listed in the document Sample functions to analyze.

2. Construct a reasonably simple function and perform a full first derivative analysis on it.

What questions do you have for your instructor?