Chapter 9: Word problems *Section 3*

Optimization problems

What you need to know already:	***What you can learn here:***
➤ All differentiation rules. ➤ How to perform a full first derivative analysis.	➤ How to use derivatives and the first derivative analysis to identify the best quantitative way to perform a certain task.

One important aspect of a first derivative analysis consists of identifying any maxima or minima. When the function in question represents a concrete quantity related to an applied setting, this corresponds to finding the "*best*" or "*worst*" values that the quantity can achieve, depending on the context. For instance, if $p = f(x)$ represents profit p as a function of price x, the first derivative analysis would provide a suggestion for how to set the price so as to obtain maximum profit. Or, if the function provides the time needed to perform a task in terms of some other variable, a minimum gives us the fastest way to do it.

Identifying this kind of optimal solutions for a problem is called – you guessed it – an *optimization* problem. Here is a slightly more formal description that may help you distinguish between an optimization problem and other types of problems, thus enabling you to use the appropriate methods.

Quick portrait of an Optimization problem

An ***optimization*** problem is a word problem in which:

- ***Two quantities*** are related, one of them (dependent) being a function of the other (independent).
- The goal is to ***identify*** the value of the independent quantity that will make the dependent quantity ***largest*** or ***smallest*** within a certain acceptable range.

Since optimization problems are word problems, all the tips and methods you know about the latter apply to the former. Some tips, however, are specific to this type of problems.

Knots on your finger

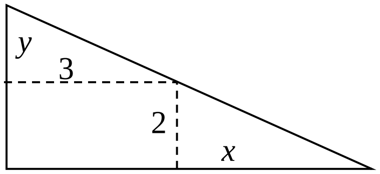
When solving an optimization problem:

- Ensure that the quantity to be optimized is expressed as a function of a ***single*** independent variable.
- If the available information leads to a relationship that involves other, auxiliary variables, find a suitable way to ***eliminate*** them by writing them in terms of the needed independent variable.
- Ensure that the optimal values found are within the practical ***restrictions*** required by the problem, that is, within the ***allowable domain*** for the function.
- Remember that a ***critical number*** may be a maximum or a minimum or neither and you need to check what it is before reaching a conclusion about the problem.
- ***Look up*** any formula you need instead of guessing.
- Use ***letters*** to denote variables, but use the actual ***numbers*** for any constants.

Not much else to say, except, once again, to encourage you to practice a lot and to reflect on your experiences as you go. Here are some examples.

Example:

A load-bearing pillar is located 2 metres North of the South wall of a room and 3 metres East of the West wall. We want to build a straight dividing wall to include the pillar so as to create a triangular storage space in the corner of the room. What dimensions would the smallest such storage space have? You may ignore the thickness of the wall.



The picture shows us the setting. We are dealing with a triangle with sides of length $3+x$ and $2+y$ respectively, so that its area is given by:

$$A=\frac{(3+x)(2+y)}{2}$$

But we need to reduce our equation to a function of a single variable and we can do that by using the similar triangles we see on the top and on the right:

$$\frac{y}{3}=\frac{2}{x} \quad \Rightarrow \quad y=\frac{6}{x}$$

Therefore, the area function is give by:

$$A=\frac{(3+x)\left(2+\frac{6}{x}\right)}{2}=\frac{(3+x)\left(\frac{2x+6}{x}\right)}{2}=\frac{(3+x)(2x+6)}{2x}$$

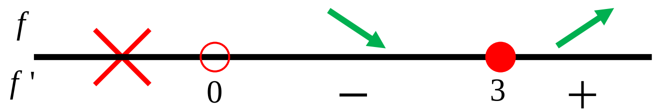
This function can be made a little simpler for the calculus steps:

$$A=\frac{2x^2+12x+18}{2x}=x+6+\frac{9}{x}$$

Now we can compute and analyze the derivative:

$$A'(x)=1-\frac{9}{x^2}=\frac{x^2-9}{x^2}$$

This is undefined at $x = 0$ and it equals 0 at $x = \pm 3$. Clearly, negative values are not allowed by our problem, so we are left with only two cut points and the following line graph:



Therefore the minimum occurs for $x = 3$.

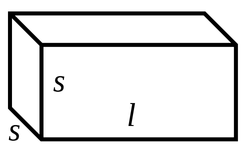
But this is not what the question asked: it asks for the dimensions of such space. By using the previous relations, we conclude that such dimensions are, respectively:

$$l = 3 + x = 6 \quad ; \quad w = 2 + y = 2 + \frac{6}{x} = 4$$

And this does indeed complete the problem.

Example:

A tissue paper box must have a volume of 144 in^3 and two of the vertical sides must be squares. If the material for the square sides costs twice as much as the rest (because of the folding and overlap), what are the dimensions of the cheapest box?



As usual, we start drawing a picture and identifying some key quantities. If *s* is the length of the edge of the square sides and *l* is the length of the remaining edges, we can consider the following quantities:

$$Area\ of\ square\ sides = 2s^2$$

$$Area\ of\ remaining\ sides = 4sl$$

But areas are not the focus of the question: cost is, so we use the cost information. We are not told how much the material costs, so let us set it to be \$*c* per square inch for the rectangular side and \$2*c* (double) for the square sides. In this way, we have that:

$$Cost\ of\ each\ square\ side = 2cs^2$$

$$Cost\ of\ each\ rectangular\ sides = csl$$

$$Total\ cost = 2\left(2cs^2\right) + 4csl = 4cs\left(s + l\right)$$

Since 4 and *c* are constant, we just need to find the minimum for the function

$$f(s) = s(s + l)$$

But we have a problem in that this formula involves both *s* and *l*, so we need to eliminate *l*. To do that we use the information about the required volume:

$$s^2 l = 144 \quad \Rightarrow \quad l = \frac{144}{s^2} \quad \Rightarrow \quad f(s) = s\left(s + \frac{144}{s^2}\right) = s^2 + \frac{144}{s}$$

To minimize this quantity, we compute its derivative:

$$f'(s) = 2s - \frac{144}{s^2} = \frac{2s^3 - 144}{s^2}$$

Of course, we need $s > 0$, or we have no box, so that the cut points are at $s = 0$ and at $s = \sqrt[3]{72} \approx 4.16$. Since the derivative is negative near 0, it follows that this latter critical value is the required minimum, corresponding to a value of $l = 144 / \sqrt[3]{72^2} \approx 8.32$.

Notice how many aspects of a word problem must be identified, examined and addressed. And this must be done completely and accurately, or the required conclusion may never be reached.

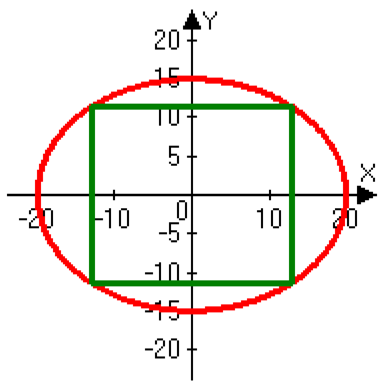
But that can be very laborious and complicated!

At the beginning more so than after having gained experience and skills. But there is no alternative!

I will now show you three more examples to provide you with more templates that can inspire you to develop your own strategy.

Example:

The strength of a rectangular beam is defined as the product of its width and the square of its height. What is the rectangular beam of maximum strength that can be cut out of an elliptical log whose major axis is 40cm and minor axis is 30cm?



Do you know how to describe mathematically an ellipse like the one described here? This is one of those steps for which some research may be needed, by calling on either your memory, a book, the web, or your instructor. One of those sources should remind you that the equation of such an ellipse, centered at the origin, as show in the graph together with the beam we need to cut, is:

$$\frac{x^2}{20^2}+\frac{y^2}{15^2}=1$$

Since we are working in the usual Cartesian plane, we can use x to represent our key variable, namely half of the width of the beam. With a beam of width $2x$, where $0<x<20$, its strength, according to the definition, will be:

$$S=2xy^2$$

The formula of the ellipse tells us that:

$$y^2=15^2\left(1-\frac{x^2}{20^2}\right)$$

Therefore the quantity to maximize is:

$$S=450x\left(1-\frac{x^2}{20^2}\right)=450x-1.125x^3$$

The derivative of this function is:

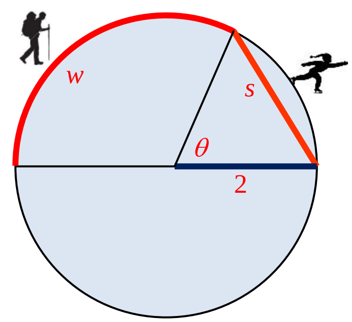
$$S'(x)=450-3.375x^2$$

The only cut points of this function are at the endpoints $x=0,\,20$ where we obviously have a minimum, as there is no thickness, and at $x=\sqrt{\frac{450}{3.375}}\approx 11.54$, which must therefore be the required maximum. The corresponding height of the beam is $y=\sqrt{15^2\left(1-\frac{x^2}{20^2}\right)}=\frac{15}{20}\sqrt{400-\frac{450}{3.375}}\approx 12.25$.

There are some situations where extra care is needed, either because a key quantity is somewhat hidden, or because what seems to be the “obvious” solution turns out to be a bad idea.

Example:

A person is standing at one point on the shore of a frozen circular lake. If the radius of the lake is 2 km, the person can walk along the shore at 3 km/hr and can skate on the ice surface at 8 km/hr, what is the fastest way to reach the opposite point on the lake?



The picture gives us a visual representation of the problem, with w being the length of the section to be walked, s the length of the section to be skated and 2 the given radius of the lake.

First of all, we can notice, from the practical nature of the problem, that if one can move faster on a direct straight line, that is probably the best choice, no? But let’s ignore that for now and go through the mathematical solution.

Can you see why I added the angle θ?

Let's think about what we need to decide: we need to figure where to stop walking and start skating. That can be done in terms of *w*, but in that case figuring out *s* is not very easy (you may want to try). Instead, I noticed that both *w* and *s* can be written in terms of θ, so I decided to use that as the independent variable.

How did I notice that? Experience and reflecting on other problems taught me that looking at a problem from a different perspective, and including quantities not mentioned in the statement of the problem can sometimes be useful and effective.

By looking up in an appropriate source, we can determine that:

$$w = R(\pi - \theta) = 2\pi - 2\theta \quad ; \quad s = 2R\sin\frac{\theta}{2} = 4\sin\frac{\theta}{2}$$

But we are trying to minimize the time needed, not the distances, so we divide each distance by the corresponding speed:

$$t_w = \frac{w}{3} = \frac{2\pi - 2\theta}{3} \quad ; \quad t_s = \frac{s}{8} = \frac{1}{2}\sin\frac{\theta}{2}$$

The total time required is therefore:

$$T = t_w + t_s = \frac{2\pi - 2\theta}{3} + \frac{1}{2}\sin\frac{\theta}{2}$$

Now we can compute the first derivative and analyze it:

$$T'(\theta) = -\frac{2}{3} + \frac{1}{4}\cos\frac{\theta}{2} = -\frac{1}{12}\left(8 - 3\cos\frac{\theta}{2}\right)$$

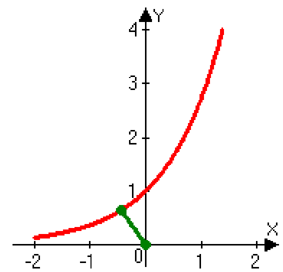
According to the general properties of the cosine function, this derivative has no critical values, as it is never 0 nor undefined! Or does it? Remember that this function came from an applied problem and therefore it has some practical domain limitations. In this case the angle cannot be less than 0 or greater than π, so the optimum solution must be at one of these values. Since the derivative is negative in between them, the minimum must occur at the lower value: $\theta = \pi$ provides the minimum and the best way to do it is by skating all the way, as we suspected initially.

Notice how the solution of an optimization problem, the best option, does not always occur in the middle. Notice also how thinking, rather than mindlessly doing computations, can sometimes be more useful and effective.

Example:

Which of the points on the graph of $y = e^x$ is closest to the origin?

If we want the closest point, we must find the point whose distance from the origin is smallest, so that this is an optimization problem.



A point on the graph of this function will have coordinates (x, e^x), so that its distance from the origin is given by the Pythagorean distance formula:

$$D(x) = \sqrt{x^2 + e^{2x}}$$

To make the problem easier, we notice that a square root is smallest when its radicand is smallest. So, we minimize the square of the distance instead:

$$SD(x) = x^2 + e^{2x}$$

We can now take its derivative and set it equal to 0:

$$SD'(x) = 2x + 2e^{2x} = 0 \quad \Rightarrow \quad x + e^{2x} = 0$$

This is not an easy equation to solve exactly, but by using an approximate method, such as Newton's method or the trace function of a calculator, we can find the approximate value of the solution to be $x = -0.43$.

Since there is only one critical value and the problem must have a minimum, this must be where it occurs. If you like, you can confirm this with a formal first derivative analysis.

Notice that in some examples I did perform a formal first derivative analysis, since there were other pieces of information that told us where the required optimum value was. When such assurance is not possible, run through the usual routine to ensure that the chosen value is, in fact, the extremum that is required.

In closing, keep in mind that while looking at other people's solution can be helpful (and there are many such solved problems on the web) the best way to learn how to do them is by…

…doing them and reflecting on the experience!

Excellent, so I trust you are ready for the *Learning questions* that are coming.

Summary

- The goal of an optimization problem is to identify the best quantitative way to complete a task.
- After the appropriate function has been constructed, the problem is solved by performing a full first derivative analysis on such function and then using it to identify the required maximum or minimum.

Common errors to avoid

- If there are practical limitations on the domain of the function to be optimized, the corresponding endpoints must always be considered as cut points in the first derivative analysis: don't forget to include them!

Learning questions for Section D 9-3

Review questions:

1. Explain how to identify a word problem as being an optimization problem.
2. Identify two potential sources of errors in the solution of an optimization problem and describe how to avoid them.

Memory questions:

1. Which feature identifies a word problem as being an optimization problem?
2. Which portion of the strategy to sketch the graph of a function is key to optimization problems?

Computation questions:

1. Find a positive number such that the sum of its double and its reciprocal is as small as possible.

2. Find the two non-negative numbers whose sum is 1 and whose sum of squares is as large as possible.

3. Find the positive number such that the sum of the number and the reciprocal of its square root is as small as possible.

4. Find the two non-negative numbers whose sum is 1 and the sum of whose square roots is as small as possible.

5. Find the equation of the line through the point (3, 5) whose portion in the first quadrant has the shortest length. (Hint: don't call on Pythagoras)

6. Find the dimensions of the largest triangle formed by the *x*-axis, the *y*-axis and a line though the point (3, 2).

7. Where is the slope of the function $f\left(x\right)=\dfrac{5}{x^2+2}$ maximum? Where is it minimum? Where is it smallest?

8. An isosceles triangle has a perimeter of 4 units. What should the length of its equal sides be so that the area is the largest possible?

9. It is well known that the largest rectangle that can be inscribed in a unit circle is a square. This square leaves four curved regions inside the circle, as shown in the picture. Determine the dimensions of the largest rectangle that can be inscribed in one of these left-over regions.

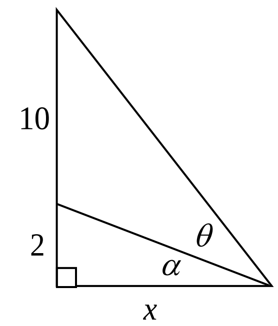
10. A circular sector has a perimeter of 4 units. What should the length of its radius be so that the area is the largest possible?

11. Determine the dimensions of the largest rectangle that that can be inscribed within the region of the first quadrant bounded by the curve $y=3-\cosh x$. Perform a full first derivative analysis to identify the solution and determine the required critical value by using Newton's method and obtaining at least 3 significant digits.

12. Which is the largest rectangle that has the upper side on the line $y=5$ and the two opposite vertices on the parabola $y=3x^2+2$? I expect you to present all the major steps needed in setting up and completely solving this problem by using calculus methods.

13. Determine the coordinates of the point P on the curve $y=\ln x$ such that the slope of the line joining P to the origin is greatest. This problem can be solved without treating it as an optimization problem, but I expect you to solve it as such.

14. In the figure shown here, what value of *x* corresponds to the largest value of θ?



15. Determine the smallest value of the constant *a* for which the graph of the function $f\left(x\right)=a^x-x$ is always above the *x* axis.

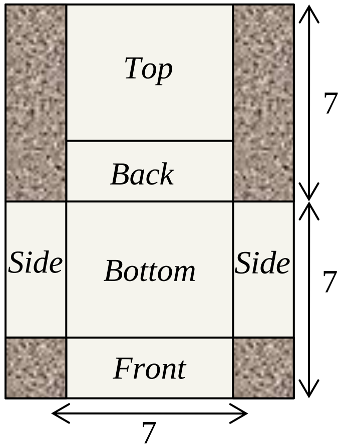
Theory questions:

1. Why is it important to consider the end points of the domain when solving an optimization problem?
2. When does the Extreme Value Theorem NOT guarantee the existence of a solution for an optimization problem?
3. Excluding the stating of the conclusion, what is the last step in completing an optimization problem?
4. In an optimization problem, do we normally use implicit or explicit differentiation?
5. What is being optimized by finding inflection points?
6. Which values should always be considered in the first derivative analysis of an optimization problem?
7. What values may provide the solution of an optimization problem if there is no x value for which the derivative is 0?
8. If an optimization problem asks you to find the point *closest* to a given point, which quantity are you supposed to optimize?
9. When trying to optimize the distance between two points, which is a good trick to use?
10. Which theorem can be used to be sure that an optimization problem has a solution?
11. Which calculus method have we used for both graphing a function and solving optimization problems?
12. To solve an optimization problem is it sufficient to find the critical values?
13. Is it possible that an optimization problem will have more than one solution?
14. Is a first derivative analysis needed for an optimization problem, or is a first derivative test sufficient?
15. What needs to be done when the function that describes the quantity to be optimized depends on two independent variables?

Application questions:

1. A juice box must contain 500 ml of liquid and have a square base. Which dimensions of the box will minimize the amount of cardboard needed to build it? Assume no waste and no overlaps for closing the box.
2. A chemical is being injected into a cleaning system so that its concentration after t minutes is $C = \dfrac{3t}{27+t^3}$. At what time is the concentration greatest and how much is it then?
3. A metal box with a square base must have a volume of 40 cubic feet and must be reinforced by welding all the edges. If the material costs $20 per square foot and the welding costs 5 per foot, what dimensions will minimize the cost of the box?

4. A rectangular piece of cardboard if folded as indicated in the picture so as to form a box. Determine the dimension of the box of maximum volume that can be obtained in this way.



5. An open garbage bin has to be built for an apartment complex, so as to have a volume of 200 cubic feet. The base of the bin must be such that one of its dimensions is twice the other. What are the dimensions of the bin that will require the least amount of material?

6. You need to build a rectangular storage container with an open top. The container has to be built of steel, two of its vertical sides must be square and its volume must be 10 m^3.
   a) Since soldering the sides together will generate much labour and expense, you are asked to build the box with least amount of soldering. What are the dimensions of such optimal box?
   b) Assuming that the material costs \$25 per square metre and soldering costs \$60 per metre, which function must be optimized to design the least expensive box?

7. An open garbage bin has to be built for an apartment complex, so as to have a volume of 200 cubic feet. The base of the bin must be such that one of its dimensions is twice the other, and it must be made of sturdier material that costs twice as much as the material used for the sides. What are the dimensions of the most economical bin satisfying these criteria?

8. The main façade of a building has to be equipped with a nice window to increase the lighting of the entrance. This window must have a total area of 2 square metres.
   a) If the window must have the shape of a Norman window (a rectangle capped by a semicircle) what dimensions will require the least amount of frame (perimetre)?
   b) If the window has to have the shape of a track oval (a rectangle with semicircles on both ends) which function must be optimized?

9. An open container must be made in the shape of a right circular cylinder, with a volume of 1 liter. What dimensions will require the least amount of material?

10. A container must be built in the shape of a right circular cylinder with an open top and must hold a volume of 8π. What dimensions will require the minimum amount of material?

11. A rectangular window must be designed with a total glass area of 0.6 m^2 and with a frame that is 10 cm wide on the left and right and 12 cm on top and bottom.
   a) Which dimensions will allow for the smallest area for the whole window, including the frame?
   b) Which dimensions will allow for the shortest total length of the frame? Assume the top and bottom extend all the way to the outer margins of the window.

12. Find the dimensions of a closed 800 m^3 cylindrical oil storage tank that can be made from the least amount of sheet metal, assuming there is no wasted metal.

13. A rectangular corral is to be enclosed with 1600 m of fencing, find the maximum area possible for the corral.

14. A metal can must be designed in the shape of a right circular cylinder that must hold a volume of 8π. If the material for the top and bottom cost twice as much as the material used for the side, what dimensions will require the minimum cost of material?

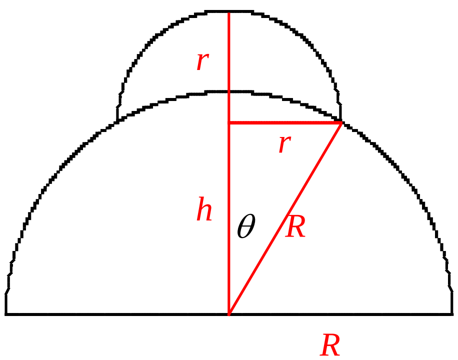
15. A metal box must have a volume of 40 cubic feet and must be made with a stronger material on top and bottom than on the sides. If the stronger material costs twice as much as the side material and if the top and bottom must be squares, what dimensions will minimize the cost of the box?

16. An alcove has to be designed for a wall of a new house. The total area occupied by it should be 0.6 m^2 and its frame should be 12 cm on the left and right sides and 14 cm on the top and bottom.
   a) Which dimensions will allow for the largest available area of the alcove?
   b) Which dimensions will allow for the least total length of frame? Assume that the left and right sides extend all the way to the top and bottom.

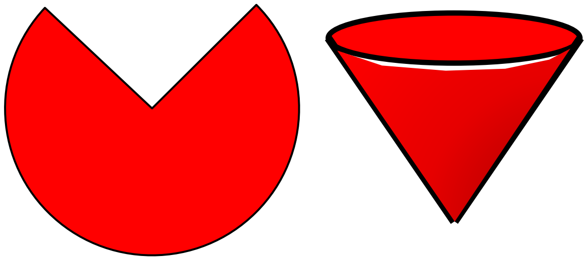
17. An open box is to be made from a square piece of cardboard whose sides are 20 cm long, from cutting equal squares from the corners and bending up the sides. Determine the size of the square that is to be cut so that the volume of the box may be a maximum

18. A power station is located at (0, 6) in a co-ordinate plane and two factories are located at (-2, 0) and (2, 0) respectively. In order to bring electric power to the two factories, a cable is run from the station straight south until a certain point (0, y), then from there straight towards each of the two factories.
   a) What value of y will allow the use of the least amount of cable?
   b) If the price of the cable going straight south is twice as much as the price of the cable going to the factories (since it has to carry twice as much power), what value of y will allow for the lowest total cost of the cable?

19. A hemispherical dome is to be placed on top of another hemispherical dome as depicted in the sectional diagram below. Assuming that the size of the bottom dome is known, determine the radius of the top hemisphere that will generate the largest height for the combined structure.



20. A cone-shaped drinking cup is made from a circular piece of paper of radius R by cutting a sector and joining the resulting edges, as suggested by the picture. Find the maximum capacity of such a cup.



21. An open square box is to be made from a plain square sheet 40 cm long by 40 cm wide. Equal squares are cut out from the four corners of the intended box and the resulting flaps are turned upwards. What are the dimensions of the box with maximum volume that can be obtained in this way?

22. A military hovercraft is sitting on the shore of a river that is 1 mile wide. The hovercraft is supposed to reach a camp that is located 3 miles downstream on the other side of the river. You need to decide the fastest route in each of these two situations:
   a) The opposite shore is smooth, so that the craft's speed on water will be 2/3 of that on shore.
   b) The opposite shore is rather rough, so that the craft's speed on land is 2/3 of that on water.

23. Based on information found in a technical journal, a broccoli grower concludes that the yield per plant (y in kg) in his fields depends on how many plants are placed per square meter (p) according to the formula $y = \dfrac{50}{p^2+25}$. How many plants should he place per square meter so that his total yield is as large as possible?

24. As a hiker starts walking across a 200-ft bridge at 6 ft/sec, a small boat passes directly beneath the centre of the bridge, moving at 8 ft/sec. At what time will the hiker and the boat be closest?

25. A closed tin box with a square base must have a volume of 32000 cm^3 and must be made with top and bottom of double thickness. Find the dimensions for the box that minimize the amount of material used.

26. An air traffic controller is monitoring two small planes flying at the same altitude. One is flying towards the control tower from the West at 120 miles per hour and is currently 5 miles away from the tower. The other plane is now on top of the tower and flying South at 150 miles per hour. In order to respect local safety rules the planes have to be always more than 4 miles from each other, or fly at different altitudes. Should the controller order one of the planes to change altitude, or will they always be farther than 4 miles from each other?

27. Your company produces certain gadgets at a price of \$5 each. Market research indicate that if those gadgets are sold at \$15 each, there will be an average of 200 sales a month, but for each dollar increase (decrease) in the price there would 8 fewer (more) sales per month. Which price would produce the highest profit?

28. A real estate office manages a 50-apartment complex. When they charge \$800/mo all units are occupied, but for each \$50 increase, on average 2 units become vacant. If each unit requires \$35/mo of maintenance, what rent amount will generate the highest profit?

29. A store owner figures that if she prices a certain item at *x* dollars, her profit will be given by the function $p(x) = -2x^2 + 12x + 5$. Present a sketch of the graph of this function in a suitable window, describe what its intercepts represent and identify one way for the owner to use this graph profitably.

30. A factory is being built on one side of a river and must be connected to a power station that is on the other side of the river and 3 km upstream. If the land cable costs \$100/km and the underwater cable costs \$150/km, how far along the shore should the land cable be run to minimize the total cable costs? Express your answer in terms of the width of the river, assumed to run straight.

31. The projection screen of a small auditorium is 10 feet high and its lower edge is 2 feet above eye level. How far should you stand from the wall so that the screen is seen in the largest possible vertical angle?

32. The *stiffness* of a rectangular beam is defined as the product of its width and the cube of its height. What are the dimensions of the stiffest beam that can be obtained from a circular log with a radius of 30 cm?

33. The efficiency *E* of a car as a function of speed *V* is given as $E = 0.768V - .00006V^2$ where V is measured in km / hour. At what speed does maximum efficiency occur?

34. The glass wall of a greenhouse forms an angle of 40° with the ground. A storage space must be created at the bottom of the wall by placing a 12’ wide board along the wall, with its bottom edge on the ground and the top edge on the wall. What angle should the board make with the ground so that the resulting space is largest?

Templated questions:

In any optimization problem you are working, identify:

1. The practical domain limitation.

2. The value of the optimized function at the optimum value.

3. The general formula needed to relate the quantities in question and the form it takes in the specific situation.

What questions do you have for your instructor?