Integration by inverse substitution by using the secant function

What you need to know already:

- How to use inverse trig substitution to integrate a function involving a $\sqrt{a^2 - b^2x^2}$ form.

What you can learn here:

- How to use inverse trig substitution to integrate a function involving a $\sqrt{b^2x^2 - a^2}$ form.

If we try the substitution $bx = a \sin \theta$ on the form $\sqrt{b^2x^2 - a^2}$ we end up with a negative under the root, which creates somewhat of a problem! We can avoid the problem by using the same method, but a different trig identity.

**Strategy for integrals that contain $\sqrt{b^2x^2 - a^2}$**

For integrands involving a factor of this form:

1. Try the inverse substitution: $bx = a \sec \theta$, $b dx = a \sec \theta \tan \theta d\theta$
2. Try to compute the new integral with suitable methods.

3. If this works, change back to the original variable, by using other trigonometric identities and/or the triangle model, as necessary.

Same issues presented in the earlier note regarding domain and extended uses apply here. All you need is practice, so here is an example.

**Example:** $\int \frac{x^2}{\sqrt{x^2 - 9}} \, dx$

We try the substitution:

$x = 3 \sec \theta$, $dx = 3 \sec \theta \tan \theta \, d\theta$, $\frac{3}{x} = \cos \theta$

This time the relation is $\frac{x}{3} = \sec \theta$, so that $x$ must represent the hypotenuse.
and 3 the adjacent side. This corresponds to the triangle representation shown here.

\[
\begin{align*}
\sqrt{x^2 - 9} & \\
\theta & \\
3 &
\end{align*}
\]

This leads to:

\[
\int \frac{x^2}{\sqrt{x^2 - 9}} \, dx = \int \frac{9 \sec^2 \theta}{\sqrt{9 \sec^2 \theta - 9}} \sec \theta \tan \theta \, d\theta
\]

\[
= \int 9 \sec^2 \theta \sec \theta \tan \theta \, d\theta = 9 \int \sec^3 \theta \, d\theta
\]

We saw earlier that the last integral above is given by:

\[
= \frac{1}{2} \left( \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right) + c
\]

By using this and the triangle representation of the substitution, remembering that secant is the reciprocal of cosine, we conclude that:

\[
= \frac{9}{2} \left( \frac{x}{3} \sqrt{x^2 - 9} + \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| \right) + c
\]

No other particular tricks to show you, so it is time for your integration calisthenics!

\[
\begin{align*}
\int \frac{2}{\sqrt{3x^2 + 12x - 15}} \, dx &= \int \frac{2}{\sqrt{3(x^2 + 4x) - 15}} \, dx \\
&= \int \frac{2}{\sqrt{3(x^2 + 4x + 4) - 15}} \, dx \\
&= \int \frac{2}{\sqrt{3(x + 2)^2 - 27}} \, dx
\end{align*}
\]

Now we make the inverse trig substitution:

\[
x + 2 = 3 \sec \theta, \, dx = 3 \sec \theta \tan \theta \, d\theta
\]

This changes our integral to:

\[
\frac{2}{\sqrt{3}} \int 3 \sec \theta \tan \theta \, d\theta = \frac{2}{\sqrt{3}} \int \sec \theta \, d\theta = \frac{2}{\sqrt{3}} \ln |\sec \theta + \tan \theta| + c
\]

Finally we switch back to \( x \):

\[
= \frac{2}{\sqrt{3}} \ln \left| \frac{x + 2}{3} + \frac{\sqrt{(x + 2)^2 + 9}}{3} \right| + c
\]
Summary

➢ To deal with integrands that contain a square root of the form \( \sqrt{b^2 x^2 - a^2} \), we use the inverse substitution \( bx = a \sec \theta, \ bdx = a \sec \theta \tan \theta d\theta \) and hope that the resulting integral can be computed.

➢ To return to the original variable, standard trigonometric identities and a suitable triangle model allow us to obtain an elegant formula for the integral.

Common errors to avoid

➢ Watch the algebra as you do the substitutions, both ways.

➢ Use the triangle model to return to the original variable, rather than constructing convoluted compositions of trig functions and their inverses.

Learning questions for Section I 2-10

Review questions:

1. Describe when and how to use the method of integration by inverse substitution by using the secant function.

Memory questions:

1. Which inverse substitution is used to integrate functions containing a factor of the form \( \sqrt{b^2 x^2 - a^2} \)?
**Computation questions:**

Evaluate the integrals proposed in questions 1-6 by using a suitable inverse substitution.

1. \( \int \frac{dx}{\sqrt{x^2 - 6x}} \)

2. \( \int \frac{x^3}{\sqrt{x^2 - 9}} \, dx \)

3. \( \int \frac{x}{\sqrt{x^2 - 4x + 1}} \, dx \)

4. \( \int \frac{x^2}{(x^2 - a^2)^{3/2}} \, dx \)

5. \( \int \frac{dx}{(2x^2 - 4x - 1)^{3/2}} \)

6. \( \int \frac{e^x \, dx}{(2e^{2x} - 4e^x - 1)^{3/2}} \)

**What questions do you have for your instructor?**