Integration by partial fractions

with non-repeated linear factors

What you need to know already:	What you can learn here:
➤ How the general method of integration by partial fractions is supposed to work.	➤ How to apply it when the denominator is a product of non-repeated linear factors.

To recap, here is the general strategy to integrate a rational function.

Knot on your finger

To integrate a rational function:

1. Put it in the form of a ***polynomial plus a proper fraction***.
2. ***Factor the denominator*** of the proper fraction completely.
3. Split the fraction by ***reversing the common denominator*** procedure.
4. ***Integrate*** the polynomial and each individual fraction by using basic integration methods.

In this section we shall focus on how to implement the third step in the simplest situation, namely when the rational function is a proper fraction and its denominator factors completely through non-repeating linear factors.

Say what?

Definition

A polynomial $b_q x^q + b_{q-1}x^{q-1} + \ldots + b_1 x + b_0$ factors as a product of ***non-repeated linear factors*** if it can be factored as:

$$(c_1 x - h_1)(c_2 x - h_2)\cdots(c_q x - h_q)$$

where no two of the roots $\frac{h_1}{c_1}, \frac{h_2}{c_2}, \cdots, \frac{h_q}{c_q}$ are equal.

Examples:

The polynomial x^2-1 factors as $(x+1)(x-1)$, therefore it factors as a product of non-repeated linear factors.

Similarly, the polynomial $3x^2+7x-6$ factors as $(3x-2)(x+3)$, so that its two roots are $\frac{2}{3}$ and -3: it is a product of non-repeated linear factors.

However, the polynomial x^2-2x+1 factors as $(x-1)(x-1)=(x-1)^2$ and is therefore NOT of this type.

Also, the polynomial x^3-1 factors as $(x-1)(x^2+x+1)$ and, since the second factor is quadratic and non-factorable, it is again NOT of this type.

Finally, the polynomial $3x^2-12x+12$ factors as $(3x-6)(x-2)$, which makes it look as two different factors. But notice that the root of both factors is 2, so that this is NOT of this type. In fact, it can be written as $3(x-2)^2$, showing that the factor is indeed repeated.

I get it! I get it! Now what?

Now we can look at how to split a proper fraction in this case, thus becoming able to integrate the corresponding function. For simplicity, I will indicate a product of factors by only writing down the first and last ones.

Strategy for splitting a proper fraction whose denominator consists of non-repeated linear factors

When a proper rational function has a denominator that can be written as a product of non-repeated linear factors of the form:

$$f(x)=\frac{a_px^p+\cdots+a_1x+a_0}{(c_1x-h_1)\cdots(c_qx-h_q)}$$

then it can be split into a sum of q fractions, each having one of the linear factors in the denominator:

$$f(x)=\frac{b_1}{c_1x-h_1}+\cdots+\frac{b_q}{c_qx-h_q}$$

for appropriate values of the coefficients b_i .

To determine the values of such coefficients:

1. ***Equate*** the original function and the desired form:
$$\frac{a_px^p+\cdots+a_1x+a_0}{(c_1x-h_1)\cdots(c_qx-h_q)}=$$
$$=\frac{b_1}{c_1x-h_1}+\cdots+\frac{b_q}{c_qx-h_q}$$

2. ***Combine*** the fractions on the right side by generating a common denominator:
$$=\frac{b_1(c_2x-h_2)\cdots(c_qx-h_q)+\cdots+b_q(c_1x-h_1)\cdots(c_{q-1}x-h_{q-1})}{(c_1x-h_1)\cdots(c_qx-h_q)}$$

3. ***Equate*** the numerator of the original function to the numerator of the single fraction so obtained:
$$a_px^p+\cdots+a_1x+a_0=$$
$$=b_1(c_2x-h_2)\cdots(c_qx-h_q)+\cdots+b_q(c_1x-h_1)\cdots(c_{q-1}x-h_{q-1})$$

4. Let $x_i = \dfrac{h_i}{c_i}, 1 \le i \le q$ to ***obtain*** the values of the corresponding b_i:

$$b_i = \frac{a_p x_i^p + \cdots + a_1 x_i + a_0}{(c_1 x_i - h_1)\cdots(c_{i-1}x_i - h_{i-1})(c_{i+1}x_i - h_{i+1})\cdots(c_q x_i - h_q)}$$

We shall call the values x_i ***smart values***, as they allow us to obtain the corresponding coefficients in a fast and effective way.

I don't feel that smart at this point! This is very complicated!

In fact, this looks horribly complicated at the beginning, but in reality it is just long. The general notation makes it obscure, so here are two examples to show you how the process works in practice:

Example: $f(x) = \dfrac{4x+1}{x^2-9}$

This function can be written as $f(x) = \dfrac{4x+1}{x^2-9} = \dfrac{4x+1}{(x-3)(x+3)}$, so we apply the procedure. We equate the given fraction to a sum of fractions, each with a single factor in the denominator and a constant on top. We can also use an easier notation for the constants:

$$\frac{4x+1}{(x-3)(x+3)} = \frac{a}{x-3} + \frac{b}{x+3}$$

Next we combine the fractions on the right side:

$$\frac{4x+1}{(x-3)(x+3)} = \frac{a(x+3)+b(x-3)}{(x-3)(x+3)}$$

Next we equate the numerators:

$$4x+1 = a(x+3)+b(x-3)$$

Then we use the *smart values* of $x_i = -3, 3$, the two roots, to find the coefficients For *b* we use the root of its denominator, which is -3:

$$x=-3 \quad \Rightarrow \quad 4(-3)+1 = a(-3+3)+b(-3-3)$$

$$\Rightarrow \quad -11 = -6b \quad \Rightarrow \quad b = \frac{11}{6}$$

For *a*, we use its root, which is 3:

$$x=3 \quad \Rightarrow \quad 4(3)+1 = a(3+3)+b(3-3)$$

$$\Rightarrow \quad 13 = 6a \quad \Rightarrow \quad a = \frac{13}{6}$$

Therefore:

$$f(x) = \frac{4x+1}{x^2-9} = \frac{13}{6}\frac{1}{x-3} + \frac{11}{6}\frac{1}{x+3}$$

At this point we are ready to integrate our function:

$$\int \frac{4x+1}{x^2-9}dx = \int \left(\frac{13}{6}\frac{1}{x-3} + \frac{11}{6}\frac{1}{x+3}\right)dx =$$

$$= \frac{13}{6}\ln|x-3| + \frac{11}{6}\ln|x+3| + c$$

Notice that the second and third steps of the procedure can be combined by simply cross multiplying the two fractions we have at that point, since the denominators are equal, just written differently. That can simplify the implementation, as shown in the next example.

Example: $f(x)=\dfrac{22x+1}{x^2-5}$

The denominator here factors as $\left(x+\sqrt{5}\right)\left(x-\sqrt{5}\right)$, so we can use our strategy. We write the given and desired forms:

$$\frac{22x+1}{x^2-5}=\frac{a}{x-\sqrt{5}}+\frac{b}{x+\sqrt{5}}$$

We now cross multiply to get:

$$22x+1=a\left(x+\sqrt{5}\right)+b\left(x-\sqrt{5}\right)$$

We now use the smart values to determine the coefficients:

$$x=\sqrt{5}\quad\Rightarrow\quad 22\sqrt{5}+1=2a\sqrt{5}\quad\Rightarrow\quad a=\frac{22\sqrt{5}+1}{2\sqrt{5}}=11+\frac{1}{2\sqrt{5}}$$

$$x=-\sqrt{5}\quad\Rightarrow\quad 1-22\sqrt{5}=-2b\sqrt{5}\quad\Rightarrow\quad b=\frac{22\sqrt{5}-1}{2\sqrt{5}}=11-\frac{1}{2\sqrt{5}}$$

Messy numbers, but simple steps, even if tedious.

We can now integrate our function.

$$\int\frac{22x+1}{x^2-5}dx=\left(11+\frac{1}{2\sqrt{5}}\right)\int\frac{dx}{x-\sqrt{5}}+\left(11-\frac{1}{2\sqrt{5}}\right)\int\frac{dx}{x+\sqrt{5}}$$

$$=\left(11+\frac{1}{2\sqrt{5}}\right)\ln\left|x-\sqrt{5}\right|+\left(11-\frac{1}{2\sqrt{5}}\right)\ln\left|x+\sqrt{5}\right|+c$$

Remember that, even though the method of partial fractions is meant *only* for rational functions, it may be possible to use it for other integrands as well, provided they can be changed into a rational function by a suitable substitution. So, here is an example that puts together everything we have seen so far.

Example: $\displaystyle\int\frac{3\ln^3x+\ln^2x+4\ln x+4}{x\left(3\ln^2x+7\ln x-6\right)}dx$

This looks very intimidating, but remember that if I ask you to compute it, I have checked that it can be done!

We start with the substitution $u=\ln x,\ xdu=dx$ to get rid of the logarithm:

$$=\int\frac{3u^3+u^2+4u+4}{3u^2+7u-6}du$$

This integrand is not a proper fraction, so, we use long division to separate a polynomial.

$$\begin{array}{r|l} & u-2 \\ \hline 3u^2+7u-6 & 3u^3\ +u^2\ +4u+4 \\ & 3u^3+7u^2-6u \\ \hline & 0\ -6u^2\ +10u+4 \\ & -6u^2-14u+12 \\ \hline & 0\quad 24u\ -8 \end{array}$$

Therefore we can write:

$$\int\frac{3u^3+u^2+4u+8}{3u^2+7u-6}du=\int(u-2)du+8\int\frac{3u-1}{3u^2+7u-6}du$$

The first integral is now easy and we focus on the second: remember the salami technique! We have seen earlier that the denominator factors as $(3u-2)(u+3)$, so we set:

$$\frac{3u-1}{3u^2+7u-6}=\frac{a}{3u-2}+\frac{b}{u+3}$$

We now cross multiply:

$$3u-1=a(u+3)+b(3u-2)$$

Next, we use the roots $\frac{2}{3}$ and -3 as smart values to find the coefficients:

$$u=\frac{2}{3} \Rightarrow 1=a\left(\frac{2}{3}+3\right) \Rightarrow a=\frac{3}{11}$$

$$u=-3 \Rightarrow -10=b(-11) \Rightarrow b=\frac{10}{11}$$

And we are now ready to arrive at the integral:

$$=\int (u-2)\,du+\frac{8}{11}\left[3\int\frac{du}{3u-2}+10\int\frac{du}{u+3}\right]$$

$$=\frac{u^2}{2}-2u+\frac{8}{11}\left[3\ln|3u-2|+10\ln|u+3|\right]+c$$

And back to the original variable:

$$=\frac{\ln^2 x}{2}-2\ln x+\frac{8}{11}\left[3\ln|3\ln x-2|+10\ln|\ln x+3|\right]+c$$

And we may even combine those logarithms, but we won't do it here: enough steps already!

Wow! That was one monster integral!

That's why we call it "*higher education!*" I hope you will practice enough to be able to do the same, eh?

But back to the key method we are discussing, there is an alternative method to find the coefficients we are after in this section. It is a method that does not generalize to what we shall do in the later sections, but if you like it, you can use it!

To figure out how it works, notice that if $f(x)=\frac{p(x)}{(c_1x-h_1)\cdots(c_nx-h_n)}$ is in proper form and has non-repeated linear factors in the denominator, we can write:

$$\frac{p(x)}{(c_1x-h_1)\cdots(c_nx-h_n)}=\frac{b_1}{(c_1x-h_1)}+\cdots+\frac{b_n}{(c_nx-h_n)}$$

If we multiply both sides by (c_1x-h_1), we get:

$$\frac{p(x)}{(c_2x-h_2)\cdots(c_nx-h_n)}=b_1+\frac{b_2(c_1x-h_1)}{(c_2x-h_2)}+\cdots+\frac{b_n(c_1x-h_1)}{(c_nx-h_n)}$$

If we take the limit of both sides as *x* approaches the smart value h_1/c_1, all fractions on the right side become 0, so that we get:

$$\lim_{x\to\frac{h_1}{c_1}}\frac{p(x)}{(c_2x-h_2)\cdots(c_nx-h_n)}=\lim_{x\to\frac{h_1}{c_1}}\left(b_1+\frac{b_2(c_1x-h_1)}{(c_2x-h_2)}+\cdots+\frac{b_n(c_1x-h_1)}{(c_nx-h_n)}\right)=b_1$$

Of course this works for the other coefficients as well, and this gives us a way to get the coefficients we are after by computing some simple limits.

OK, but the general notation is still creating some problems. Examples?

Example: $f(x)=\frac{4x+1}{x^2-9}$

We have seen that:

$$\frac{4x+1}{(x-3)(x+3)}=\frac{a}{x-3}+\frac{b}{x+3}$$

So, to find *a* and *b* we compute:

$$a=\lim_{x\to 3}\frac{(4x+1)(x-3)}{(x-3)(x+3)}=\lim_{x\to 3}\frac{4x+1}{x+3}=\frac{13}{6}$$

$$b=\lim_{x\to -3}\frac{(4x+1)(x+3)}{(x-3)(x+3)}=\lim_{x\to -3}\frac{4x+1}{x-3}=\frac{11}{6}$$

These are the same values we found earlier.

Wow! This is way faster: why didn't you tell us earlier?

As I said, this is a faster way, but cannot be generalized to the cases we'll see later. So, use it, but learn the previous method, as it will become the essential one in general! For your benefit, here is what the method looks like in the other example we saw.

Example: $f(x)=\dfrac{22x+1}{x^2-5}$

We write the desired partial fractions form as:

$$\frac{22x+1}{x^2-5}=\frac{a}{x-\sqrt{5}}+\frac{b}{x+\sqrt{5}}$$

Therefore:

$$a=\lim_{x\to\sqrt{5}}\frac{22x+1}{x+\sqrt{5}}=\frac{22\sqrt{5}+1}{2\sqrt{5}}=11+\frac{1}{2\sqrt{5}}$$

$$b=\lim_{x\to-\sqrt{5}}\frac{22x+1}{x-\sqrt{5}}=\frac{-22\sqrt{5}+1}{-2\sqrt{5}}=11-\frac{1}{2\sqrt{5}}$$

Again, a method with simpler steps, worth learning, but don't lean on them too much as the next section will deal with a different situation where this method will not work anymore!

Summary

- To split a proper rational function whose denominator is a product of non-repeated linear factors, construct one fraction for each such factor, place it in the denominator and put a different constant on the numerator of each. Then use smart values to figure out what those constants are.

Common errors to avoid

- The process is not difficult, but it is long and convoluted, so stay focused, watch for small algebra errors and don't get frustrated too easily.

Learning questions for Section I 2-14

Review questions:

1. Explain, in your own words, how to integrate a proper rational function whose denominator is a product of non-repeated linear factors.

2. Describe both methods for finding the coefficients of a partial sum decomposition when the rational function is in proper form and the denominator factors into non-repeating linear factors.

Memory questions:

1. When a proper rational function has non-repeated linear factors in the denominator, into how many fractions must it be split?

2. When a proper rational function has non-repeated linear factors in the denominator, what must go in the numerator of each smaller fraction?

Computation questions:

Evaluate the integrals proposed in questions 1-6. In all cases, use both method for finding the needed coefficients.

1. $\int \frac{x^4+2x-1}{x^3-x}\,dx$

2. $\int \frac{x^2+4}{x^2-4}\,dx$

3. $\int \frac{x+1}{x^2-4x+3}\,dx$

4. $\int \frac{x^3-x+3}{x^2+x-2}\,dx$

5. $\int \frac{2x^4-5x^2+x+3}{2x^2-8}\,dx$.

6. $\int \frac{x-1}{x^2+4x+3}\,dx$

7. $\int \frac{\sin x\cos x}{\sin^2 x-4\sin x+3}\,dx$

8. $\int \frac{x^{-1}\ln x}{\ln^2 x-4\ln x+3}\,dx$

Proof questions:

1. Prove that $\int \sec x\,dx = \ln\left|\sec x+\tan x\right|+c$ without using the "trick" that gave us the answer earlier, but instead changing the integral to $\int \sec x\,dx = \int \frac{1}{\cos x}\,dx = \int \frac{\cos x}{\cos^2 x}\,dx$ and then using trig identities, a substitution, partial fractions and properties of logarithms to arrive at the answer.

Templated questions:

1. Construct a simple rational function whose denominator factors into non-repeated linear factors and integrate it.

What questions do you have for your instructor?