Chapter 2: Integration methods *Section 16*

Integration by partial fractions

The general case

What you need to know already:	*What you can learn here:*
➤ How to apply the method of partial fractions when the denominator is a product of non-repeated linear or quadratic factors.	➤ How to apply the method of partial fractions when the denominator is a product of both linear and quadratic factors, or when some factor is raised to a power.

So, we now know how to integrate a rational function whose denominator factors into either non-repeated linear factors only, or as non-repeated irreducible quadratic factors only. What do we do when that denominator factors into a mixture of linear and quadratic factors and perhaps some of them are raised to a power? Well, I hope you will be glad to know that the method is not much different, although, as you can imagine, its implementation will be longer.

Strategy for splitting a proper fraction whose denominator consists of mixture or power of linear and irreducible quadratic factors

Once the denominator of a rational function has been factored into linear and irreducible quadratic factors, we split into partial fractions as follows:

1. We assume that the original function can be written as a ***sum of simple fractions***, each of which has either a linear or an irreducible quadratic factor in the denominator.
2. For each fraction with a ***linear factor*** in the denominator, we place an unknown ***constant*** in the numerator.
3. For each fraction with a ***quadratic factor*** in the denominator, we place an undetermined ***linear function*** in the numerator.
4. If a factor is present to the power of n, we assign to it ***n fractions***, each with the same denominator, but in increasing powers, and each with a ***different*** numerator of the appropriate type.

5. We ***determine all coefficients*** by using smart and/or easy values, as possible.
6. We ***integrate*** each fraction as appropriate.

I can see that this is going to get ugly!

Actually, the description above, once again, is probably worse than its implementation, especially since I will use simple cases in all examples, exercises and tests. More complex ones are more sanely obtained by using a computer algebra system that has been programmed to implement the procedure.

I will conclude with a few more examples. Obviously, there are infinitely many possible ones, so plenty of chance for you to practice!

Example: $\int \frac{22x+1}{x^3-8}dx$

The denominator is a difference of cubes that factors as:

$$x^3-8=(x-2)(x^2+2x+4)$$

So, we write:

$$\frac{22x+1}{x^3-8}=\frac{a}{x-2}+\frac{bx+c}{x^2+2x+4}$$

By cross multiplying, this gives us:

$$22x+1=a(x^2+2x+4)+(bx+c)(x-2)$$

We have one smart value, $x=2$, and we can pick two easy values, say $x=0,1$. This leads us to:

$$x=2 \;\Rightarrow\; 45=a(12) \;\Rightarrow\; a=\frac{15}{4}$$

$$x=0 \;\Rightarrow\; 1=4a-2c \;\Rightarrow\; c=\frac{4a-1}{2}=7$$

$$x=1 \;\Rightarrow\; 23=7\frac{15}{4}-(b+7) \;\Rightarrow\; b=\frac{105}{4}-30=-\frac{15}{4}$$

Therefore, we can write:

$$\int\frac{22x+1}{x^3-8}dx=\int\frac{15}{4(x-2)}dx+\frac{1}{4}\int\frac{-15x+28}{x^2+2x+4}dx$$

The first integral is easy:

$$\int\frac{15}{4(x-2)}dx=\frac{15}{4}\ln|x-2|+c$$

For the second, the derivative of the denominator is $2x+2$, so we adjust the numerator (to acquire that) and complete the square in preparation for the next step (that comes from experience!):

$$\frac{1}{4}\int\frac{28-15x}{x^2+2x+4}dx=\frac{1}{8}\int\frac{56-30x}{(x+1)^2+3}dx=\frac{1}{8}\int\frac{86-30-30x}{(x+1)^2+3}dx$$

$$=\frac{43}{4}\int\frac{1}{(x+1)^2+3}dx-\frac{15}{8}\int\frac{2+2x}{(x+1)^2+3}dx$$

We are now ready for the substitutions: $u=\sqrt{3}(x+1)$ for the first and $v=(x+1)^2+3$ for the second. This leads to:

$$=\frac{43}{4\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)-\frac{15}{8}\ln\left((x+1)^2+3\right)+c$$

To conclude:

$$\int\frac{22x+1}{x^3-8}dx=\frac{15}{4}\ln|x-2|+\frac{43}{4\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)-\frac{15}{8}\ln\left((x+1)^2+3\right)+c$$

Example: $\int\frac{x}{(x^2+3x+2)(x+2)}dx$

We begin by noticing that:

$$\frac{x}{(x^2+3x+2)(x+2)} = \frac{x}{(x+1)(x+2)(x+2)}$$
$$= \frac{x}{(x+1)(x+2)^2}$$

We have one linear, non-repeated factor, namely $(x+1)$, and another linear factor repeated twice, that is, raised to the second power, namely $(x+2)^2$. Therefore, we need three fractions, one for the first and two for the second, with increasing powers, each with a constant in the numerator:

$$\frac{x}{(x+1)(x+2)^2} = \frac{a}{x+1} + \frac{b}{x+2} + \frac{c}{(x+2)^2}$$

Now we are ready to complete the process. We first combine the three fractions and equate the two numerators:

$$x = a(x+2)^2 + b(x+1)(x+2) + c(x+1)$$

Then we use the two available smart values –1 and –2 to get two of the coefficients:

$$x=-1 \;\Rightarrow\; -1=a$$
$$x=-2 \;\Rightarrow\; -2=-c \;\Rightarrow\; c=2$$

We now need an easy value to complete the task. The value $x=0$ is still available and we can use the values of *a* and *c* that we have already found:

$$x=0 \;\Rightarrow\; 0=-4+2b+2 \;\Rightarrow\; b=1$$

We can now split our function as:

$$\int \frac{x}{(x^2+3x+2)(x+2)}dx = \int\left(\frac{-1}{x+1}+\frac{1}{x+2}+\frac{2}{(x+2)^2}\right)dx$$

and basic integration provides the conclusion:

$$\int \frac{x}{(x^2+3x+2)(x+2)}dx = -\ln|x+1| + \ln|x+2| - \frac{2}{x+2} + c$$

Example: $\displaystyle\int \frac{2x^3-2x^2-8x+2}{(x-2)^2(x^2+2)}dx$

The integrand is a rational function in proper form, so we proceed with the partial fraction decomposition. We need two fractions for the linear factor, with increasing power and a constant on top, and one fraction with the quadratic factor in the bottom and a linear expression on top:

$$\frac{2x^3-2x^2-8x+2}{(x-2)^2(x^2+2)} = \frac{a}{x-2} + \frac{b}{(x-2)^2} + \frac{cx+d}{x^2+2}$$

We merge the three fractions through a common denominator and equate the numerators:

$$2x^3-2x^2-8x+2=$$
$$=a(x-2)(x^2+2)+b(x^2+2)+(cx+d)(x-2)^2$$

We only have one smart value, namely $x=2$, from which we get:

$$-6=6b \;\Rightarrow\; b=-1$$

We now use some easy values:

$$x=0 \;\Rightarrow\; 2=-4a-2+4d \;\Rightarrow\; d=a+1$$
$$x=2 \;\Rightarrow\; -6=6b \;\Rightarrow\; b=-1$$
$$x=1 \;\Rightarrow\; -4=-3a-3+(c+a+1) \;\Rightarrow\; c=2a-4$$
$$x=-1 \;\Rightarrow\; 6=-9a-3+(4-2a+a+1)9 \;\Rightarrow\; a=2$$
$$\Rightarrow\; d=3,\, c=0$$

Hence, we can write:

$$\int \frac{2x^3-2x^2-8x+2}{(x-2)^2(x^2+2)}dx = \int\frac{2}{x-2}dx - \int\frac{1}{(x-2)^2}dx + \int\frac{3}{x^2+2}dx$$
$$= 2\ln|x-2| + \frac{1}{x-2} + \frac{3}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + c$$

Example: $\int \frac{x^2}{(4+x^2)^2}dx$

This time we have a single, but repeated quadratic factor, so we need two fractions with that factor in the bottom, with increasing power, and two linear functions on top:

$$\frac{x^2}{(4+x^2)^2} = \frac{ax+b}{(4+x^2)} + \frac{cx+d}{(4+x^2)^2}$$

We cross multiply and use easy values, since no smart ones are available:

$$x^2 = (ax+b)(4+x^2) + cx + d$$

$$x=0 \quad \Rightarrow \quad 0 = 4b + d$$

$$x=1 \quad \Rightarrow \quad 1 = 5(a+b) + c + d \quad \Rightarrow \quad 1 = 5a + 5b + c + d$$

$$x=-1 \quad \Rightarrow \quad 1 = 5(-a+b) - c + d \quad \Rightarrow \quad 1 = -5a + 5b - c + d$$

$$x=2 \quad \Rightarrow \quad 4 = 8(2a+b) + 2c + d \quad \Rightarrow \quad 4 = 16a + 8b + 2c + d$$

Solving the system consisting of these four equations leads to:

$$a = -\frac{1}{4},\ b = 0, c = \frac{1}{4}, d = 0$$

Therefore:

$$\frac{x^2}{(4+x^2)^2} = -\frac{1}{4}\left(\frac{x}{(4+x^2)} + \frac{x}{(4+x^2)^2}\right)$$

The substitution $u = 4 + x^2$ works for both fractions and allows us to conclude that:

$$\int \frac{x^2}{(4+x^2)^2}dx = \frac{1}{8}\left(\ln\left(4+x^2\right) - \frac{1}{4+x^2}\right) + c$$

As you can imagine, there are umpteen other possible combinations, but they all will require the same general method, so I leave them to your discovery in your practice work.

Summary

- When linear and quadratic factors both appear in the denominator, we split the function by allowing for both, each with its own required numerator.
- If a factor is raised to a power, we need one fraction for each such power, with denominators of increasing power and numerators of the same type, but with different coefficients.

Common errors to avoid

- Don’t give up too soon! These integrals tend to be long and full of details, but can be tamed, with a little patience.
- Keep your work on the page nicely organized, or you may lose or change pieces as you go, just because of poor writing or reading.

Learning questions for Section I 2-16

Review questions:

1. Explain how the method of integration by partial fractions works in the general case.

Memory questions:

1. When splitting a proper ration functional into partial fractions, how many fractions must be built for a denominator factor that is repeated *n* times?

Computation questions:

Evaluate the integrals proposed in question 1-25.

1. $\int \frac{x+1}{x^5-x}dx$

2. $\int \frac{3x+5}{(x^2+3x+2)(x+1)}dx$

3. $\int \frac{x^3+2x^2+2x}{x^3+2x^2+3x}dx$

4. $\int \frac{2x-3}{(x-1)^2}dx$

5. $\int \frac{x+15}{x^2-8x+16}dx$

6. $\int \frac{2x^3-4x-8}{x^4-x^3+4x^2-4x}dx$

7. $\int \frac{x+15}{x^3-8x^2+16x}dx$

8. $\int \frac{7+6x-x^2}{(x-1)(x^2+2x+3)}dx$

9. $\int \frac{x+1}{x^3+2x^2+2x}dx$

10. $\int \frac{x+1}{x^2+4x+3}dx$

11. $\int \frac{3x^2-x+2}{(x^2+2)(x-2)}dx$

12. $\int \frac{3x^2-4x+5}{(x-1)(x^2+1)}dx$

13. $\int \frac{2x^3-5x^2-8x-4}{x^2(x+1)^2}dx$

14. $\int \frac{x^4}{x^3-1}dx$

15. $\int \frac{x^2+2x-1}{x^3+x}dx$

16. $\int \frac{x^3-2x^2+3x-1}{x^5+2x^3+x}\,dx$

17. $\int \frac{5x^2-x+6}{x^3+2x}\,dx$

18. $\int \frac{x-1}{x^2+4x+3}\,dx$

19. $\int \frac{4x^2+2x-1}{x^3+x^2}\,dx$

20. $\int \frac{8x^3-4x^2+8x}{(x^2-1)(x^2+1)}\,dx$

21. $\int \frac{3+4x-x^2}{(x-1)(x^2+1)}\,dx$

22. $\int \frac{2x^2+3x+2}{x^3+x}\,dx$

23. $\int \frac{3x^2+2}{x^4+2x^2}\,dx$

24. $\int \frac{dt}{e^t-e^{-t}}$

25. $\int \frac{2}{3+\sqrt[4]{x}}\,dx$

Theory questions:

1. In general, when does the method needed by the main question not work?

2. Which two transcendental functions can be part of the integral of a rational function?

3. Which partial fractions decomposition is needed for a rational function of the form $y=\frac{P(x)}{\left(x^2+1\right)\left(x^2-1\right)}$?

4. Which partial fractions decomposition is needed for a rational function of the form $y=\frac{Q(x)}{x^3\left(x^2-1\right)}$?

5. Which algebraic method is needed as the first step to integrate a function of the form $\frac{1}{ax^2+bx+c}$ if the denominator is irreducible?

6. What is the connection between the method of partial fractions and linear algebra?

Proof questions:

1. Determine a general formula for integrals of the type $\int \frac{dx}{x^2\left(x+k\right)}$, where $k \neq 0$.

Templated questions:

1. Construct a simple rational function and integrate it.

What questions do you have for your instructor?