Integration by partial fractions
The general case

What you need to know already:

How to apply the method of partial fractions when the denominator is a product of non-repeated linear or quadratic factors.

What you can learn here:

How to apply the method of partial fractions when the denominator is a product of both linear and quadratic factors, or when some factor is raised to a power.

So, we now know how to integrate a rational function whose denominator factors into either non-repeated linear factors only, or as non-repeated irreducible quadratic factors only. What do we do when that denominator factors into a mixture of linear and quadratic factors and perhaps some of them are raised to a power? Well, I hope you will be glad to know that the method is not much different, although, as you can imagine, its implementation will be longer.

Strategy for splitting a proper fraction whose denominator consists of mixture or power of linear and irreducible quadratic factors

Once the denominator of a rational function has been factored into linear and irreducible quadratic factors, we split into partial fractions as follows:

1. We assume that the original function can be written as a sum of simple fractions, each of which has either a linear or an irreducible quadratic factor in the denominator.
2. For each fraction with a linear factor in the denominator, we place an unknown constant in the numerator.
3. For each fraction with a quadratic factor in the denominator, we place an undetermined linear function in the numerator.
4. If a factor is present to the power of n, we assign to it n fractions, each with the same denominator, but in increasing powers, and each with a different numerator of the appropriate type.
5. We **determine all coefficients** by using smart and/or easy values, as possible.

6. We **integrate** each fraction as appropriate.

---

I can see that this is going to get ugly!

Actually, the description above, once again, is probably worse than its implementation, especially since I will use simple cases in all examples, exercises and tests. More complex ones are more sanely obtained by using a computer algebra system that has been programmed to implement the procedure.

I will conclude with a few more examples. Obviously, there are infinitely many possible ones, so plenty of chance for you to practice!

**Example:**

\[
\int \frac{22x + 1}{x^3 - 8} \, dx
\]

The denominator is a difference of cubes that factors as:

\[
x^3 - 8 = (x - 2)(x^2 + 2x + 4)
\]

So, we write:

\[
\frac{22x + 1}{x^3 - 8} = \frac{a}{x - 2} + \frac{bx + c}{x^2 + 2x + 4}
\]

By cross multiplying, this gives us:

\[
22x + 1 = a(x^2 + 2x + 4) + (bx + c)(x - 2)
\]

We have one smart value, \(x = 2\), and we can pick two easy values, say \(x = 0, 1\). This leads us to:

\[
x = 2 \implies 45 = a(12) \implies a = \frac{15}{4}
\]

\[
x = 0 \implies 1 = 4a - 2c \implies c = \frac{4a - 1}{2} = 7
\]

Therefore, we can write:

\[
\int \frac{22x + 1}{x^3 - 8} \, dx = \frac{15}{4} \int \frac{1}{x - 2} \, dx + \frac{1}{4} \int -15x + 28 \, dx
\]

The first integral is easy:

\[
\int \frac{15}{4(x - 2)} \, dx = \frac{15}{4} \ln|x - 2| + c
\]

For the second, the derivative of the denominator is \(2x + 2\), so we adjust the numerator (to acquire that) and complete the square in preparation for the next step (that comes from experience!):

\[
\frac{1}{4} \int \frac{28 - 15x}{x^2 + 2x + 4} \, dx = \frac{1}{8} \int \frac{56 - 30x}{(x + 1)^2 + 3} \, dx = \frac{1}{8} \int \frac{86 - 30x}{(x + 1)^2 + 3} \, dx
\]

\[
= \frac{43}{4} \int \frac{1}{(x + 1)^2 + 3} \, dx - \frac{15}{8} \int \frac{2 + 2x}{(x + 1)^2 + 3} \, dx
\]

We are now ready for the substitutions: \(u = \sqrt{3}(x + 1)\) for the first and \(v = (x + 1)^2 + 3\) for the second. This leads to:

\[
= \frac{43}{4\sqrt{3}} \tan^{-1}\left(\frac{x + 1}{\sqrt{3}}\right) - \frac{15}{8} \ln((x + 1)^2 + 3) + c
\]

To conclude:

\[
\int \frac{22x + 1}{x^3 - 8} \, dx = \frac{15}{4} \ln|x - 2| + \frac{43}{4\sqrt{3}} \tan^{-1}\left(\frac{x + 1}{\sqrt{3}}\right) - \frac{15}{8} \ln((x + 1)^2 + 3) + c
\]

**Example:**

\[
\int \frac{x}{(x^2 + 3x + 2)(x + 2)} \, dx
\]

We begin by noticing that:
\[
\frac{x}{(x^2 + 3x + 2)(x + 2)} = \frac{x}{(x + 1)(x + 2)(x + 2)}
\]

We have one linear, non-repeated factor, namely \((x + 1)\), and another linear factor repeated twice, that is, raised to the second power, namely \((x + 2)^2\). Therefore, we need three fractions, one for the first and two for the second, with increasing powers, each with a constant in the numerator:

\[
\frac{x}{(x + 1)(x + 2)^2} = \frac{a}{x + 1} + \frac{b}{x + 2} + \frac{c}{(x + 2)^2}
\]

Now we are ready to complete the process. We first combine the three fractions and equate the two numerators:

\[
\frac{2}{x + 1} + \frac{2}{x + 2} + \frac{2}{(x + 2)^2} = \frac{x}{x + 2}
\]

Then we use the two available smart values \(-1\) and \(-2\) to get two of the coefficients:

\[
x = -1 \quad \Rightarrow \quad -1 = a
\]

\[
x = -2 \quad \Rightarrow \quad 2 = -c \quad \Rightarrow \quad c = 2
\]

We now need an easy value to complete the task. The value \(x = 0\) is still available and we can use the values of \(a\) and \(c\) that we have already found:

\[
x = 0 \quad \Rightarrow \quad 0 = -4 + 2b + 2 \quad \Rightarrow \quad b = 1
\]

We can now split our function as:

\[
\int \frac{x}{(x^2 + 3x + 2)(x + 2)} \, dx = \int \left( \frac{-1}{x + 1} + \frac{1}{x + 2} + \frac{2}{(x + 2)^2} \right) \, dx
\]

and basic integration provides the conclusion:

\[
\int \frac{x}{(x^2 + 3x + 2)(x + 2)} \, dx = -\ln|x + 1| + \ln|x + 2| - \frac{2}{x + 2} + c
\]

**Example:**

\[
\int \frac{2x^3 - 2x^2 - 8x + 2}{(x - 2)^2(x^2 + 2)} \, dx
\]

The integrand is a rational function in proper form, so we proceed with the partial fraction decomposition. We need two fractions for the linear factor, with increasing power and a constant on top, and one fraction with the quadratic factor in the bottom and a linear expression on top:

\[
\frac{2x^3 - 2x^2 - 8x + 2}{(x - 2)^2(x^2 + 2)} = \frac{a}{x - 2} + \frac{b}{(x - 2)^2} + \frac{cx + d}{x^2 + 2}
\]

We merge the three fractions through a common denominator and equate the numerators:

\[
2x^3 - 2x^2 - 8x + 2 = a(x - 2)(x^2 + 2) + b(x - 2)^2 + (cx + d)(x - 2)^2
\]

We only have one smart value, namely \(x = 2\), from which we get:

\[
-6 = 6b \quad \Rightarrow \quad b = -1
\]

We now use some easy values:

\[
x = 0 \quad \Rightarrow \quad 2 = -4a - 2 + 4d \quad \Rightarrow \quad d = a + 1
\]

\[
x = 2 \quad \Rightarrow \quad -6 = 6b \quad \Rightarrow \quad b = -1
\]

\[
x = 1 \quad \Rightarrow \quad -4 = -3a - 3 + (c + a + 1) \quad \Rightarrow \quad c = 2a - 4
\]

\[
x = -1 \quad \Rightarrow \quad 6 = -9a - 3 + (4 - 2a + a + 1) \quad \Rightarrow \quad a = 2
\]

\[
\Rightarrow \quad d = 3, c = 0
\]

Hence, we can write:

\[
\int \frac{2x^3 - 2x^2 - 8x + 2}{(x - 2)^2(x^2 + 2)} \, dx = \int \frac{2}{x - 2} \, dx + \int \frac{1}{(x - 2)^2} \, dx + \int \frac{3}{x^2 + 2} \, dx
\]

\[
= 2 \ln|x - 2| + \frac{1}{x - 2} + \frac{3}{\sqrt{2}} \tan^{-1}\frac{x}{\sqrt{2}} + c
\]
Example: $$\int \frac{x^2}{(4 + x^2)^2} \, dx$$

This time we have a single, but repeated quadratic factor, so we need two fractions with that factor in the bottom, with increasing power, and two linear functions on top:

$$\frac{x^2}{(4 + x^2)^2} = \frac{ax + b}{(4 + x^2)} + \frac{cx + d}{(4 + x^2)^2}$$

We cross multiply and use easy values, since no smart ones are available:

$$x = 0 \implies 0 = 4b + d$$
$$x = 1 \implies 1 = 5(a + b) + c + d \implies 1 = 5a + 5b + c + d$$
$$x = -1 \implies 1 = 5(-a + b) - c + d \implies 1 = -5a + 5b - c + d$$
$$x = 2 \implies 4 = 8(2a + b) + 2c + d \implies 4 = 16a + 8b + 2c + d$$

Solving the system consisting of these four equations leads to:

$$a = -\frac{1}{4}, \, b = 0, \, c = \frac{1}{4}, \, d = 0$$

Therefore:

$$\int \frac{x^2}{(4 + x^2)^2} \, dx = -\frac{1}{4} \left( \frac{x}{4 + x^2} + \frac{x}{(4 + x^2)^2} \right)$$

The substitution $$u = 4 + x^2$$ works for both fractions and allows us to conclude that:

$$\int \frac{x^2}{(4 + x^2)^2} \, dx = \frac{1}{8} \left( \ln(4 + x^2) - \frac{1}{4 + x^2} \right) + c$$

As you can imagine, there are umpteen other possible combinations, but they all will require the same general method, so I leave them to your discovery in your practice work.

**Summary**

- When linear and quadratic factors both appear in the denominator, we split the function by allowing for both, each with its own required numerator.
- If a factor is raised to a power, we need one fraction for each such power, with denominators of increasing power and numerators of the same type, but with different coefficients.

**Common errors to avoid**

- Don’t give up too soon! These integrals tend to be long and full of details, but can be tamed, with a little patience.
- Keep your work on the page nicely organized, or you may lose or change pieces as you go, just because of poor writing or reading.
Learning questions for Section I 2-16

Review questions:

1. Explain how the method of integration by partial fractions works in the general case.

Memory questions:

1. When splitting a proper rational function into partial fractions, how many fractions must be built for a denominator factor that is repeated \( n \) times?

Computation questions:

Evaluate the integrals proposed in question 1-25.

1. \( \int \frac{x+1}{x^2-x} \, dx \)
2. \( \int \frac{3x+5}{(x^2+3x+2)(x+1)} \, dx \)
3. \( \int \frac{x^3+2x^2+2x}{x^3+2x^2+3x} \, dx \)
4. \( \int \frac{2x-3}{(x-1)^2} \, dx \)
5. \( \int \frac{x+15}{x^3-8x+16} \, dx \)

6. \( \int \frac{2x^3-4x-8}{x^4-x^3+4x^2-4x} \, dx \)
7. \( \int \frac{x+15}{x^3-8x^2+16x} \, dx \)
8. \( \int \frac{7+6x-x^2}{(x-1)(x^2+2x+3)} \, dx \)
9. \( \int \frac{x+1}{x^3+2x^2+2x} \, dx \)
10. \( \int \frac{x+1}{x^2+4x+3} \, dx \)
11. \( \int \frac{3x^2-x+2}{(x^2+2)(x-2)} \, dx \)
12. \( \int \frac{3x^2-4x+5}{(x-1)(x^2+1)} \, dx \)
13. \( \int \frac{2x^3-5x^2-8x-4}{x^2(x+1)^2} \, dx \)
14. \( \int \frac{x^4}{x^3-1} \, dx \)
15. \( \int \frac{x^2+2x-1}{x^3+x} \, dx \)
### Theory questions:

1. In general, when does the method needed by the main question not work?

2. Which two transcendental functions can be part of the integral of a rational function?

3. Which partial fractions decomposition is needed for a rational function of the form \( y = \frac{P(x)}{(x^2 + 1)(x^2 - 1)} \)?

4. Which partial fractions decomposition is needed for a rational function of the form \( y = \frac{Q(x)}{x^3(x^2 - 1)} \)?

5. Which algebraic method is needed as the first step to integrate a function of the form \( \frac{1}{ax^2 + bx + c} \) if the denominator is irreducible?

6. What is the connection between the method of partial fractions and linear algebra?
Proof questions:

1. Determine a general formula for integrals of the type \( \int \frac{dx}{x^2(x + k)} \), where \( k \neq 0 \).

Templated questions:

1. Construct a simple rational function and integrate it.

What questions do you have for your instructor?