**Chapter 2: Integration methods** **Section 4**

# Integration by parts with a single visible factor

| What you need to know already: | What you can learn here: |
|---|---|
| ➤ Integration by parts with two visible factors. | ➤ How to apply integration by parts to certain integrals consisting of a single factor. |

What is the general antiderivative of $y = \ln x$ ?

*Have we seen it yet?*

No, but would you believe that we can compute it through integration by parts?

*But how? The formula for integration by parts requires two factors and here we only have one!*

Well, we are going to make a second one appear out of thin air: mathematical magic! Watch!

**Example:** $\int \ln x \, dx$

Abracadabra, or, as a more modern version would have it, *apparate*!

$$\int \ln x \, dx = \int 1 \times \ln x \, dx$$

Now we can set:

$$f(x) = \ln x, \; g'(x) = 1 \quad \Rightarrow \quad f'(x) = \frac{1}{x}, \; g(x) = x$$

This leads to:

$$\int \ln x \, dx = x \ln x - \int 1 \, dx = x \ln x - x + c$$

And finally:

$$\int \ln x \, dx = x \ln x - x + c$$

Keep this integration formula in mind, since it will be useful and used later.

*I was expecting something more spectacular!*

Then stay tuned, because more spectacular tricks will be seen soon. Here we just used the basic facts that any quantity multiplied by 1 does not change and that the function $y = 1$ is simple to integrate.

## Strategy for applying integration by parts to a single factor

If the function $f(x)$ is ***easy to differentiate***, we can try to integrate it by parts by using $g'(x) = 1$ .

This strategy may not work always, but it does work in many useful situations, and in particular to integrate basic ***inverse functions***.

*Well, the logarithm is an inverse function, right?*

Yes, and so is the function of the next example.

***Example:*** $\int \sin^{-1} x\, dx$

We use the same method and set:

$$f(x) = \sin^{-1} x,\ g'(x) = 1 \quad \Rightarrow \quad f'(x) = \frac{1}{\sqrt{1-x^2}},\ g(x) = x$$

This leads to:

$$\int \sin^{-1} x\, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx$$

Is this an easier integral? Yes, since we can use a substitution for it:

$$u = 1 - x^2 \quad \Rightarrow \quad du = -2x dx$$

$$\Rightarrow \quad -\int \frac{x}{\sqrt{1-x^2}}\, dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\, du = \frac{1}{2}\int u^{-1/2} du = \sqrt{1-x^2} + c$$

Therefore we can conclude that:

$$\int \sin^{-1} x\, dx = x \sin^{-1} x + \sqrt{1-x^2} + c$$

*This one is pretty ugly!*

Don't forget that integration is an inverse operation and therefore it often produces strange looking conclusions. But the fact remains that this method is rather effective for single factors.

Since we are dealing with a single factor, the method does not change any when applying it to other situations. So, time for your practice!

## Summary

- A function consisting of a single factor may be integrated by parts by using it as the function to differentiate and $g'(x) = 1$ as the one to integrate.
- This method works quite nicely to integrate basic inverse functions.

## Common errors to avoid

- Don't forget that a factor of 1 can be considered as part of any product.

# *Learning questions for Section I 2-4*

## ***Review questions:***

1. Describe when and how to use the method of integration by parts when the integrand consists of a single factor.

## ***Memory questions:***

1. What is the indefinite integral of $f(x) = \ln x$?

2. If we want to apply the method of integration by parts to an integral of the form $\int f(x)dx$ what do we let *g'(x)* be?

## ***Computation questions:***

Compute the integrals proposed in questions 1-6.

1. $\int \cos^{-1} x dx$

2. $\int \tan^{-1} x dx$

3. $\int \sinh^{-1} x dx$

4. $\int \cosh^{-1} x dx$

5. $\int \tanh^{-1} x dx$

6. $\int (\ln x)^2 \, dx$

7. Evaluate the integral $\int \ln \sqrt{x} dx$ by using first a suitable substitution, and then again by using integration by parts with a single factor.

8. Compute $\int \sqrt{x} \, dx$ by using the method of this section (after all, a square root is an inverse function) and check that you obtain the same conclusion that the power rule would give.

9. Integration by parts can be used in three different ways to evaluate the integral $\int x \ln x \, dx$. Use all such ways and check that they give the same answer.

10. In some situations, the factor of 1 that we have used in this sections can be introduced as a ratio of two equal quantities that allow us to rewrite the integral in a workable form. This requires additional experience, but some such instances are well known. For instance, to compute $\int \frac{x^2}{(x\sin x + \cos x)^2}dx$ no substitution will work and the derivative of the integrand is way worse than the original. But, if we multiply and divide by $\cos x$, we can write it as $\int \frac{x}{\cos x}\frac{x\cos x}{(x\sin x + \cos x)^2}dx$. Now we have a product of two visible factors: your task is to complete the work!

## *Proof questions:*

1. If we apply integration by parts to $\int \frac{1}{x}dx$ with $f'(x)=1, g(x)=\frac{1}{x}$ we get:

$$f(x)=x, g'(x)=-\frac{1}{x^2} \quad \Rightarrow \quad \int \frac{1}{x}dx = 1 + \int \frac{1}{x}dx.$$

But if we cancel the same integral on both sides we can conclude that 0=1! How can this be?

## *Application questions:*

1. Two cars are waiting at an intersection with a 4-way stop sign. One is a Ford and is coming from the East and needs to turn North; the other is a Honda coming from the South and needs to turn West. Since they will not interfere with each other, they start at the same time. If the Ford moves with an acceleration of $\frac{3}{t+1}\,m/sec^2$ and the Honda with one of $\frac{2}{t+1}\,m/sec^2$, how fast will they move away from each other after 5 seconds? Ignore the time needed to turn and the space to be covered at the intersection, don't worry about getting a numerical value from the exact values, and feel free to use substitution to simplify the notation.

## *What questions do you have for your instructor?*