Chapter 2: Integration methods — **Section 6**

Reduction formulae through integration by parts

What you need to know already:	What you can learn here:
➤ How integration by parts works.	➤ How integration by parts can generate formulae to decrease the power of the integrand to a lower value.

The usual examples and exercises of integration usually involve low powers of basic functions. That is done to decrease the level of difficulty of the computations and focus instead on how the relevant methods work.

However, in some real applications the power of the integrand may be fairly high and integration by parts may have to be used many times before arriving at the final expression for the general antiderivative. When this is the case, it may be useful to find a shortcut to simplify the repetitions of the steps.

Definition

A ***reduction formula*** is a formula that expresses the integral of a function involving a generic power in terms of another integral with a similar structure, but involving a lower power.

Constructing a reduction formula allows us to compute integrals involving large powers of the variable by applying that formula repeatedly, without computing a single integral, until the last integral is easy to obtain through some basic formula or method.

Example: $\int x^n e^x dx$

This integral involves the n-th power of x, where n can be an integer number as high as we want. The structure of the integrand as a product of two functions of different types suggests that integration by parts may be useful, so we try it:

$$f' = e^x,\ g = x^n,\ f = e^x,\ g' = nx^{n-1}$$

This leads to:

$$\int x^n e^x dx = fg - \int g' f dx = x^n e^x - n\int x^{n-1} e^x dx$$

We can therefore conclude that for any value of n:

$$\int x^n e^x dx = x^n e^x - n\int x^{n-1} e^x dx$$

This is a reduction formula, since it does not give us an explicit formula for the original integral, but reduces it to a formula that involves the same integral, but with a lower power of x.

For instance, we can apply this formula repeatedly to compute $\int x^3 e^x dx$.

The first use gives:

$$\int x^3 e^x dx = x^3 e^x - 3\int x^2 e^x dx$$

Now we apply it again to the remaining integral:

$$x^3 e^x - 3\int x^2 e^x dx = x^3 e^x - 3\left(x^2 e^x - 2\int x e^x dx\right)$$
$$= x^3 e^x - 3x^2 e^x + 6\int x e^x dx$$

And one last time:

$$x^3 e^x - 3x^2 e^x + 6\left(x e^x - \int e^x dx\right) = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + c$$

Example: $\int x^n \cos x \, dx$

We have again a situation suitable for an iterative reduction formula, so we try it with the obvious choice of parts:

$$f(x) = x^n, g'(x) = \cos x \quad \Rightarrow \quad f'(x) = nx^{n-1}, g(x) = \sin x$$
$$\int x^n \cos x \, dx = x^n \sin x - n\int x^{n-1} \sin x \, dx$$

This time we get a new integral that has a similar structure, but not quite the same, since it involves the sine instead of the cosine. But we know that the integral of the sine involves the cosine, so we try it again.

$$f(x) = x^{n-1}, g'(x) = \sin x$$
$$\Rightarrow \quad f'(x) = (n-1)x^{n-2}, g(x) = -\cos x$$
$$x^n \sin x - n\int x^{n-1} \sin x \, dx =$$
$$= x^n \sin x - n\left(-x^{n-1}\cos x + (n-1)\int x^{n-2} \cos x \, dx\right)$$

This produces the reduction formula:

$$\int x^n \cos x \, dx =$$
$$= x^n \sin x + nx^{n-1}\cos x - n(n-1)\int x^{n-2} \cos x \, dx$$

We can use this formula to compute $\int x^4 \cos x \, dx$. The first time we get:

$$\int x^4 \cos x \, dx = x^4 \sin x + 4x^3 \cos x - 12\int x^2 \cos x \, dx$$

We apply it again to get:

$$\int x^4 \cos x \, dx =$$
$$= x^4 \sin x + 4x^3 \cos x - 12\left(x^2 \sin x + 2x\cos x - 2\int \cos x \, dx\right)$$
$$= x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x - 24x\cos x + 24\sin x + c$$

Summary

- A power reduction formula allows us to change an integral involving a high power to one involving a lower power.
- Power reduction formulae are often obtained by using integration by parts.

Common errors to avoid

- Watch the algebra steps, as they can become rather long and complex!

Learning questions for Section I 2-6

Review questions:

1. Describe what a reduction formula is and how it is used.

Computation questions:

1. Construct a reduction formula for integrals of the form $\int x^n e^{-2x} dx$ and use it to determine the antiderivative of $x^2 e^{-2x}$.

2. Construct the iterative reduction formula for $\int x^n \sin x \, dx$.

3. Use the iterative reduction formula for $\int x^n \cos x \, dx$ to compute $\int x^5 \cos x \, dx$.

Theory questions:

1. The formula $\int u^n \cos u\,du = u^n \sin u - n\int u^{n-1} \sin u\,du$ is an example of what type of integration formulae?

What questions do you have for your instructor?