Chapter 3: Basics of differential equations *Section 1*

Introduction to ODE's

What you need to know already:	*What you can learn here:*
➤ All basic concepts and methods of differentiation and indefinite integration.	➤ What an Ordinary Differential Equation (ODE) is, why ODE's are important and what basic jargon is associated with them.

In the last chapter we have seen several methods for finding the indefinite integral of a function and we also saw some examples in which we found a specific antiderivative. Both problems are special cases of a more general mathematical problem to which we shall devote our attention in this chapter. I will start with the relevant definitions.

Definition

A ***differential equation*** is an equation that includes one or more derivatives of a function and, possibly, its dependent and independent variables.

To ***solve*** a differential equation means to find all functions that make that equation true.

Example: $y' = -\frac{1}{x^2}$

This equation can be considered as a differential equation, since it includes the derivative of some $y = f(x)$. It also includes *x*, but does not include *y*. The presence of either is not necessary for the definition.

Notice that solving this differential equation simply means finding all functions for which the equation is true, that is, the indefinite integral:

$$\int -\frac{1}{x^2}\,dx$$

Therefore we know that the solutions of this equation are given by:

$$y = \begin{cases} \frac{1}{x} + c_1 & if\ \ x < 0 \\ \frac{1}{x} + c_2 & if\ \ x > 0 \end{cases}$$

Example: $y' + x^2 = y$

This is also a differential equation and involves both *x* and *y*. Do we know how to solve this equation now? No, since we cannot have both variables in the integral we may want to consider. However, by the end of this chapter, you will have learned how to solve it ☺.

Why do we need to know how to solve differential equations?

Great question! Because in many important applied problems, both in science and in other areas, we do not have enough empirical information to identify the formula for a quantity directly, but we may have enough information about its rate of change to construct a differential equation involving it. Therefore, knowing how to solve differential equations allows us to identify the quantity itself and thus solve the practical problem that involves it. Here are three examples.

Example:

If an object falls towards the surface of a planet, the height y of the object is determined by the forces acting on it, such as gravity, atmospheric friction, electromagnetic fields and so on. That means that such height is given by:

$$y''(t) = a(y(t))$$

where $a(y(t))$ is some function depending on height itself and/or time on, depending on the forces present. But this is a differential equation, so that solving it may allow us to know the height of the object at any time, thus allowing us to guide and control its movement.

This looks like rocket science to me!

It is! Maybe it is the first such example for you, but I hope it will not be the last!

Example:

If you have an ideal pendulum (straight rope and no air friction), it is fairly simple to see that the angle θ formed by the pendulum with a vertical line, the constant length l of the pendulum and gravity g are related by the equation:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

This is a differential equation, as it involves a function θ of time and its (second) derivative. Solving it may allow us to design and/or control mechanisms based on the pendulum.

Example:

And now for something rather different!

If two armies are poised for war, their strength (measured as some combination of size and of amount of equipment) are given by x and y, and their efficiency (ability to inflict damage) is given by constants a and b respectively, we can see that:

$$\frac{dy}{dt} = -bx, \quad \frac{dx}{dt} = -ay$$

This is telling us that each army decreases at a rate proportional to the strength of the other. If we combine these two equations we obtain:

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{bx}{ay} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{bx}{ay}$$

This is again a differential equation and its solution may give the army commanders essential information about whether to continue fighting or negotiate a surrender!

As you can see from these examples, differential equations come in all kinds and shapes and they continue to be a major area in mathematics. In fact, if you meet someone who says he (or she) is an "*applied mathematician*", that does not necessarily mean that they work for industry; more likely they are academics who study differential equations.

However, we shall limit this introduction to a very small set of such equations and we shall only have the opportunity to see a very limited variety of methods of solutions. Therefore, consider this just as a way to whet your appetite!

And here is very basic and important terminology. Think of it as an appetizer!

Definition

An ***ordinary differential equation (ODE)*** is an equation involving a function $y = f(x)$ of ***one variable*** only.

Are you saying that there are differential equations involving functions of several variables and partial derivatives?

Absolutely! And guess what they are called?

Extraordinary differential equations?

No: since they involve partial derivatives, they are called *partial differential equations* (PDE). They are extremely important and interesting, but they are complicated enough that you shall have to wait for a future course to learn more about them.

Here is a very important word associated with ODE's.

Definition

The ***order*** of an ODE is the order of the highest derivative it contains.

Example: $y'\sin x - y\cos x = 2e^{2x}\sin^2 x$

This is an ODE, since it involves a function $y = f(x)$ of a single variable and its derivative. Its order is 1, since the only derivative that appears is the first one.

To solve it, we need to find all functions $y = f(x)$ that make the equation true.

We can see that the function $y = e^{2x}\sin x$ is a solution, since

$$y' = 2e^{2x}\sin x + e^{2x}\cos x$$

and therefore:

$$y'\sin x - y\cos x =$$
$$= \left(2e^{2x}\sin x + e^{2x}\cos x\right)\sin x - \left(e^{2x}\sin x\right)\cos x$$
$$= 2e^{2x}\sin^2 x$$

Wait a minute: how did you come up with that particular solution? And are there more? Can we find them all?

Great questions! You are starting to think like a mathematician!

We shall see how to solve this particular ODE in section I3-5 and the issue of how to find *ALL* solutions will be foremost in our mind throughout the chapter.

However, in the practical applications I mentioned earlier, we are not interested in finding all solutions of the ODE, but only the one that solves the applied problem we have in mind. In that case an additional piece of information is needed to allow us to determine the solution we need. This leads to one more definition.

Definition

An ***initial value problem*** is an ODE together with additional conditions that are satisfied by only some, or even just one of the solutions.

To ***solve*** an initial value problem means to find the function(s) that satisfies both the differential equation and the additional condition(s).

In particular, there are many problems where the independent variable is time. In that case we may not know how a quantity changes in time, but we may know how fast that quantity changes and how much it is at some particular instant. From there we can construct the ODE and then solve it.

Example:

Consider an object that travels along the *y* axis, starting from the origin and so that its speed follows the equation

$$y'\sin t - y\cos t = 2e^{2t}\sin^2 t$$

If we want to find the position function of the object we have to solve the initial value problem consisting of the ODE and the information about where it is at time $t=0$.

I showed in the previous example that $y = e^{2t}\sin t$ is a solution of this ODE and you can see that at $t=0$ (starting time) this function places the object at the origin.

Therefore $y = e^{2t}\sin t$ is a solution of the initial value problem. Is this the only solution? We'll have to develop some methods for solving ODE's before we can answer that question.

You are almost ready to start learning how to solve some ODE's, all you need is some practice with the terminology you have seen so far.

Summary

- An ordinary differential equation (ODE) is an equation involving a function of a single variable.
- Solving an ODE means finding all the functions that make it true.
- An initial value problem requires us to find a special solution of an ODE that satisfies additional conditions.

Common errors to avoid

- Don't overestimate the importance of jargon: what special words related to ODE's did I NOT include in the *Summary*?

Learning questions for Section I 3-1

Review questions:

1. Explain what a differential equation is.
2. Explain what an initial value problem is.
3. Describe what the acronym ODE stands for and what its first word refers to.
4. Explain what the order of an ODE is.

Memory questions:

1. The presence of what element in an equation makes it a *differentia*l equation?
2. What does "ODE" stand for?
3. To what does the qualifier *"ordinary"* refer in an ordinary differential equation?
4. What is the *"order"* of a differential equation?
5. When solving a differential equation, what are we looking for?
6. If you are asked to find a function that solves a differential equation and contains a given point, what kind of problem are you solving?

Computation questions:

1. Is the function $y = -x^4 + 5x - 7$ one of the solutions of the differential equation $y' + 4x^3 = 5$?
2. Is the function $y = \sin^2 x$ one of the solutions the differential equation $\frac{y''}{2} - \sin^2 x = \cos^2 x$?

Solve the differential equations provided in questions 3-10.

3. $x^2 - y' = \sec^2 x$
4. $y'' + e^x = \cos x$
5. $y'\sqrt{x} - x^2 = x + 1.$
6. $y' \sec x - \ln x = \sqrt{x}$
7. $\frac{(1+x^2)y'+1}{1+x^2} = \frac{1}{\sqrt{1-x^2}}$

8. $\frac{y'+1}{x}=\frac{x+1}{\sqrt{1-x^2}}$

9. $y'+4x^3=5$

10. $y'-2x^2=2x+1$

Solve the initial value problems provided in questions 11-15.

11. $y'=5-4x^3,\ y(1)=5$

12. $y'=x^2-\sec^2 x,\ y(0)=3$

13. $y'+e^x=\cos x,\ y(0)=5$

14. $y'+\ln x=x^2+2,\ y(1)=e$

15. $y'=\frac{\sin^3 x}{\left(\cos^2 x+1\right)\left(\cos x+1\right)},\ \ y\left(\frac{\pi}{2}\right)=3$

16. Determine the values of *k* for which the function $y=e^{kx}$ is a solution of the ODE $y''+4y=4y'$ and check that for each such value of *k*, every function of the form $y=ae^{kx}+bxe^{kx}$ is also a solution of the ODE.

17. Determine if the parametric curve $\left(e^t, t^2-t\right)$ satisfies the ODE $xy'+1=2\ln^2 x-2y$

Theory questions:

1. What does the "O" in ODE tell us?

2. How many solutions does an ODE have in general?

3. What is added to an ODE to make it into an initial value problem?

4. If you are asked to provide the general solution of an ODE, what should the answer look like?

5. In mathematical terms, what do we call a statement like "*The growth rate is twice as big as the size?*"

6. In what way are the solution of a differential equation and the solution of an initial value problem different?

Proof questions:

1. Show that $y = \sin x$ is a solution of the ODE $y'' + y = 0$.

2. Show that $y = e^{x^2} + x$ is a solution of the ODE $y' - 2x^2 = 2xy + 1$.

3. Show that $y = ke^t$ is a solution of the ODE $y' - y = 0$ for any value of *k*. Does this imply that there are no other solutions?

4. Show that $y = 3e^t$ is a solution of the initial value problem $y' - y = 0,\ y(0) = 3$.

5. Show that any function of the form $y = \dfrac{C}{e^{\cos x}}$, where *C* is a real number, is a solution of the ODE $y' - y\sin x = 0$ and find the specific solution that contains the point $(0,1)$.

Application questions:

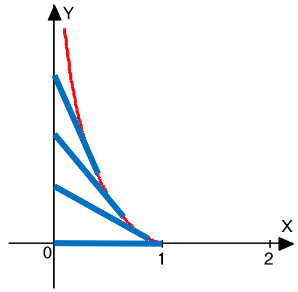
1. Explain in what sense the ODE $P' = kP$ represents the growth in time (*t*) of a population (*P*).

2. Construct an ODE representing the motion of an object falling under both gravity and the effect of air friction, the latter being proportional to the velocity of the object.

3. A certain quantity *Q* is known to change in time at a rate that depends on its size through the function $\dfrac{dQ}{dt} = 2Q^3 - 17Q^2 + 8Q$. Determine the size of this quantity for which the quantity will:
   a) keep increasing,
   b) keep decreasing
   c) remain in equilibrium.

4. Check that a population growing according to the ODE $P' = kP\left(1 - \dfrac{P}{M}\right)$ will grow fairly fast if $P \ll M$, will change slowly if $P \approx M$ and will decrease if $P > M$.

5. Imagine a buoy floating at position $(1, 0)$ on the coordinate plane and a person dragging it with a taut rope, starting from the origin and moving up along the *y* axis. Ignoring a number of complicating physical phenomena, we can assume that the buoy will follow and trace a trajectory with the property that at any time the rope is tangent to such trajectory. The idea is illustrated by this picture, where the red curve is the trajectory and the blue lines represent the rope at different times.



Prove that the trajectory is a solution of the differential equation $\dfrac{dy}{dx} = -\dfrac{\sqrt{1-x^2}}{x}$ and then use integration to obtain the explicit formula for such solution. This is a classical problem, called the *tractrix* problem, which is discussed on several web pages.

6. According to Newton’s law of cooling the rate at which an object’s temperature changes is proportional to the difference in temperature with the surrounding environment. Translate this statement into the corresponding differential equation.

Templated questions:

1. For any ODE you see, explain what makes it an ODE and identify its order.

What questions do you have for your instructor?