Special separable ODE's

What you need to know already:	*What you can learn here:*
➤ How to solve a separable ODE.	➤ The names, solutions and uses of certain special separable ODE's.

The exponential model provides the simplest applied example of a separable ODE, but there are many more. In this section you shall see a few more of them, two as examples and the rest as learning questions. We shall begin with a common population model that addresses a key deficiency of the exponential model.

Remember that the exponential growth model assumes no limitations on the eventual size of the changing quantity. In many situations this is not realistic. Instead, as the quantity grows, its capacity to grow may diminish. This is true for biological populations (where a limited supply of food may create a slow down) as well as in financial applications (where an excessive increase in the money supply creates inflation and the slowing down of its worth) and others.

In these situations a different ODE may be more suitable, one that incorporates this slowing down effect. The following model provides one such alternative.

Definition

Assume that the size $P(t)$ of a population is a function of time and can grow at a rate proportional to itself, but that practical ***limitations*** keep P always below a maximum value M, called the ***carrying capacity*** of the population.

Assume also that the same limitations ***slow down*** the rate of growth, as the size of the population increases.

In that case P may be well approximated by the ***logistic model*** ODE:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

A basic analysis of the equation that defines the logistic model reveals the following, rather realistic properties.

Technical fact

If $P(t)$ follows a logistic model with carrying capacity M, then:

- If $P \ll M$, the population grows ***almost*** as it would in an ***exponential model***.

- If $P \approx M$ and $P < M$, the ***rate of growth*** becomes ***very small***, approaching 0. This means that the population becomes stable in size, approaching the carrying capacity.
- If $P = M$ the population will not grow, but ***remain stable*** at that value. Remember that this is an approximate model!
- If $P > M$, the rate of growth becomes negative and the size of the ***population will decrease*** towards the carrying capacity.

Since the ODE of the logistic model does not include the independent variable *t*, it is clearly separable and, therefore, we know how to get its solution.

Technical fact

The logistic ODE $\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$ is separable and its general solution is given by:

$$y = \frac{M}{1 + Ae^{-kt}}$$

where A is related to the arbitrary constant coming from the integration and is given by the formula:

$$A = \frac{M - y(0)}{y(0)}$$

Proof

We separate the variables, meaning that we just leave the constants on the right side:

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right) = ky\left(\frac{M-y}{M}\right) \quad \Rightarrow \quad \frac{dy}{y(M-y)} = \frac{k}{M}dt$$

To integrate the left side, we use partial fractions:

$$\int \frac{dy}{y(M-y)} = \frac{1}{M}\int\left(\frac{1}{y} + \frac{1}{M-y}\right)dy = \frac{1}{M}\ln\left|\frac{y}{M-y}\right| + c$$

This tells us that:

$$\frac{1}{M}\ln\left|\frac{y}{M-y}\right| = \frac{k}{M}t + c \quad \Leftrightarrow \quad \ln\left|\frac{y}{M-y}\right| = kt + C$$

where $C = cM$ also represents an arbitrary constant. Because of that, we can continue to denote it as *c*. This implies that:

$$\Rightarrow \quad \frac{y}{M-y} = \pm e^{kt}e^{c} \quad \Leftrightarrow \quad \frac{y}{M-y} = Ce^{kt}$$

Here we use *C* to represent $\pm e^{c}$, which is again an arbitrary constant, so we continue to call it c, even though its value keeps changing at all these steps!

Notice also that, since $y < M$ (the maximum possible value), in an applied problem this constant is always positive.

Now we solve the equation for *y*:

$$\Rightarrow \quad y = ce^{kt}(M-y) \quad \Rightarrow \quad y = \frac{ce^{kt}M}{1 + ce^{kt}}$$

This is now an explicit solution, but it is not clear what the meaning of this final constant *c* is. To obtain a more meaningful expression, we divide top and bottom by ce^{kt}:

$$\Rightarrow \quad y = \frac{M}{1 + c^{-1}e^{-kt}} = \frac{M}{1 + Ae^{-kt}}$$

This does not look clearer than before, until we analyze what that constant A, which equals the old c^{-1} is. By evaluating this function at the initial time $t=0$ we get:

$$y(0)=\frac{M}{1+A} \quad \Rightarrow \quad A=\frac{M-y(0)}{y(0)}$$

As claimed above. Notice that this means that:

$$M=y(0)+Ay(0)$$

which provides us a concrete interpretation of this constant.

Wait a minute: how is that more meaningful or concrete? I still don't see what A represents!

Knot on your finger

The constant A in the solution of the logistic model ODE represents the ***additional*** (hence A) ***multiplier*** of the initial size (that is, $Ay(0)$) that we need to add to such size to reach the carrying capacity.

Alternatively, if we ***add A times the initial size to the initial size***, we get M.

It may not be very simple or clear, but it is a concrete interpretation! Let us look at an example.

Example:

The population of Canada in 1981 was approximately 25 million, while in 2001 it was approximately 30 million.

If we assume the growth to be exponential, its model can be determined by assuming 1981 as the starting year (t=0), so that:

$$P(t)=25e^{kt} \quad \Rightarrow \quad P(20)=30=25e^{20k}$$

$$\Rightarrow \quad k=\frac{\ln 1.25}{20}\approx 0.009$$

This means a growth rate slightly lower than 1% and a model given by:

$$P_e(t)\approx 25e^{0.009t} \quad ; \quad t=0\leftrightarrow 1981$$

According to this model, by 2081 the number of Canadians, in millions, will be:

$$P_e(100)\approx 25e^{0.009\times 100}=61.47$$

If we assume that the maximum possible population for Canada is 100 million, we can use a logistic model and we can add 3 times the original population before reaching the maximum: $25+3\times 25=100$. Therefore $A=3$ and the model is:

$$P=\frac{M}{1+Ae^{-kt}}=\frac{100}{1+3e^{-kt}}$$

However, the value of k is now different, so we re-compute it:

$$30=\frac{100}{1+3e^{-20k}} \quad \Rightarrow \quad e^{-20k}=\frac{7}{9} \quad \Rightarrow \quad k\approx 0.013$$

Notice that the value of k is now larger, since we are using the same increase over the first 20 years, but the limitation given by the carrying capacity would slow the process. Therefore, the revised estimate for 2081 is now:

$$P(100)\approx\frac{100}{1+3e^{-1.3}}=55.02$$

The growth has slowed down, as expected.

Please keep in mind that all these values are based on assumptions that are likely incorrect, approximate and changing in time. But they may be sufficient for some degree of planning.

You may want to repeat the computation by assuming a different carrying capacity, say 200 million, and compare to the actual number, although you will have to wait a while for that: long life to you!

There are other ODE's that have properties similar to the logistic model and are used to construct predictive models of populations based on different assumptions. Here are some of them.

Definition

When a size P of a population has a carrying capacity M, its size at time t may be well approximated by the ***Gompertz model*** ODE:

$$P' = kP\ln\frac{M}{P} = kP(\ln M - \ln P)$$

Definition

When the size P of a population cannot grow beyond a carrying capacity M (maximum) or shrink below a critical size m (minimum), such size may be well described by the ***window of opportunity*** model:

$$P' = kP\left(1-\frac{P}{M}\right)\left(1-\frac{m}{P}\right)$$

Both of these models simply assume a growth based on the size. But some times the size of the population can grow according to additional factors, such as time itself, based on different considerations.

Here is the most common such model.

Definition

When the size P of a population grows according to a rate that also changes periodically, such size may be well described by the ***seasonal model***:

$$P' = kP\cos rt$$

Notice that in this model the rate alternates between positive and negative values, thus generating a periodic increase and decrease of the population size.

And then there are some models that can be used effectively for apocalyptic science fiction movies!

Definition

Let c be an arbitrarily small positive number and consider the ODE:

$$\frac{dy}{dt} = ky^{1+c}$$

This seems to provide a population growth model very similar to the exponential model, only with a slightly bigger growth rate.

In fact, this is called the ***doomsday model***.

Doomsday? As in the end of the world?

Yes, and you will discover the reason for this name in the learning questions.

Please remember that all these models are based on assumptions that may or may not be valid and on estimated values that are likely to change over time, so take them with a good amount of salt!

And speaking of salt, the second type of separable ODE's that we shall look at describes the amount of salt, or any other substance, that is present in a solution that is being constantly changed and instantly mixed. More specifically, here is the setting.

Definition

In a ***mixing problem***, we are given information about the initial amount of solute present in a ***solution*** and the rate at which more solute or solution is being added or subtracted. Moreover, in order to simplify the computations, we also assume that any additional solution is ***homogenized instantly***,

By using such information, we can construct a ***separable*** ODE whose solution can be obtained by the method of separation of variables and represents the ***amount of solute*** present at any time.

Yes, I know: you need some examples to clarify what we are talking about!

Example:

A tank contains 1000 litre of water with 15 kilograms of salt dissolved in it. Brine consisting of 100 grams of salt per 10 litres of water is continuously added to the tank at the rate of 10 litres per minute and instantly mixed. To keep the tank from overflowing, solution is drained from the bottom at the same rate. Determine the function describing how much salt is in the tank at any time.

If $S(t)$ denotes the amount of salt at time *t*, our information tells us that:

$$\frac{dS}{dt} = Rate\ in - Rate\ out = \frac{0.1kg}{10L}\frac{10L}{1m} - \frac{Skg}{1000L}\frac{10L}{1m}$$

Therefore, the ODE is:

$$\frac{dS}{dt} = 0.1 - \frac{S}{100}$$

This ODE is separable, so that:

$$\frac{dS}{10-S} = \frac{dt}{100} \quad \Rightarrow \quad -\ln|10-S| = \frac{t}{100} + c$$

By using the initial value of 15 *kg*, we get:

$$-\ln|10-15| = c \quad \Rightarrow \quad c = -\ln 5 \quad \Rightarrow \quad \ln(S-10) = \ln 5 - \frac{t}{100}$$

We finally apply exponentials to get the explicit solution:

$$S - 10 = e^{\ln 5 - \frac{t}{100}} = e^{\ln 5} e^{-\frac{t}{100}} = 5e^{-\frac{t}{100}} \quad \Rightarrow \quad S = 10 + 5e^{-\frac{t}{100}}$$

Notice that eventually, that is after a very long time, the amount of salt in the tank will have the same concentration as the brine we are putting in:

$$\lim_{t\to\infty} S = \lim_{t\to\infty}\left(10 + 5e^{-\frac{t}{100}}\right) = 10\ kg$$

and 10 *kg* in 1000 *L* corresponds to 0.1 *kg/L*.

Aargh! Another type of word problem!

Of course! ODE's are very useful in practice, so you must learn how to apply them. I hope you still remember all the basic ideas about solving word problems. If you don't, it's time to refresh your memory!

Summary

- Alternative and more realistic model to describe the size of a growing population exist and are used.
- Many of these models are associated to a separable ODE and can therefore be solved as such.
- Mixing problems, in which the amount of solute in a solution is monitored, are also represented by separable ODE's.

Common errors to avoid

- Remember that all ODE models that represent practical processes are approximate and as such provide only estimates of the real process.

Learning questions for Section I 3-4

Review questions:

1. Describe what a logistic model ODE is and when it is used.
2. Describe what a mixing problem is and how it is solved.
3. Describe another type of applied separable ODE and explain when and how it is used.

Memory questions:

1. What is the general form of the ODE for a *logistic* model?
2. What does the solution of a *mixing problem* represent?
3. What type of ODE is associated with the *logistic*, *Gompertz* and *seasonal growth* models of a population?

Computation questions:

1. Find the general solution of the ODE associated to the *doomsday* model $\frac{dy}{dt} = ky^{1+c}$.

2. Find the general solution of the seasonal ODE model $P' = kP\cos rP$.

3. Assume that a certain quantity is known to change according to the ODE $\frac{dQ}{dt} = 0.2Q\cos^2(\pi t)$. Find the formula relating the quantity to time.

4. Determine the general solution of the Gompertz ODE $y' = ky\ln\frac{M}{y}$.

Theory questions:

1. Which integration method is needed to obtain the solution of a logistic ODE?

2. What happens to a logistic process if something happens that makes the carrying capacity infinite?

3. If *y* is much smaller than *M*, will a solution of the Gompertz ODE grow faster or slower than the corresponding solution to the logistic model?

4. What key assumption differentiates an exponential from a logistic model?

5. Provide the name of a population growth model that is not exponential, but whose ODE is still separable.

6. Identify two reasons why the models presented in this section provide only approximate solutions.

7. Provide an interpretation of the constant *A* that is part of the solution of a logistic ODE in terms of the initial information about the process.

Proof questions:

1. Show that any solution of the Gompertz ODE $\frac{dy}{dx} = ky\ln\frac{M}{y}$ is such that $\lim_{x\to\infty} y(x) = M$.

2. Show that in the Gompertz model, the smaller the population is, the slower it grows.

3. Show that if the size of a population follows the Gompertz model $P' = kP\ln\frac{M}{P}$, then:
   a) If $P \approx M$, $P < M$, the rate of growth becomes very small, approaching 0. What does that mean for the population?
   b) If $P = M$ the population will not grow, but remain stable at that value.

c) If $P > M$, the rate of growth becomes negative and the size of the population will decrease towards the carrying capacity.
d) If $P \ll M$, the rate of growth also becomes very small, as its limit approaches 0. What does that mean for the population?

4. Determine the general solution of the window of opportunity model $P' = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$.

5. Show that a population whose size changes according to the window of opportunity model $P' = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$ will:
   a) remain between m and *M* if it starts there
   b) become extinct if it starts below *m*
   c) decrease towards *M* if it starts above it.

6. Determine the general solution of the seasonal model $P' = kP\cos rt$.

7. The solution of a logistic model ODE depends on the initial value, the carrying capacity and the proportionality constant. Assuming that the initial value and the proportionality constant are given, discuss how the solution changes as a function of the carrying capacity, both in terms of size and rate. .

8. Prove that the solution of the logistical model with carrying capacity 1 and initial size 0.5 is the mean between the hyperbolic tangent and 1.

9. Use the general solution of the doomsday model $\frac{dy}{dt} = ky^{1+c}$ (one of the computational questions) to show if a population grows according to it, will become infinite in a finite amount of time. That finite time is the doomsday to which the name refers.

10. A growth model that is currently being used in some areas of health sciences, such as for tumor growth, uses the flip side of the doomsday model, namely the ODE $y' = ky^{\alpha}$ with $0 < \alpha < 1$. Solve this ODE and identify some of its features that may be relevant to the application in question. Graphing the solution for some values of α may be useful in your analysis.

Application questions:

1. The company for which you work has developed a new product and has marketed it for the last two months. In January it sold 200 items, while 225 were sold in February. A market analysis shows that the North-American market could sell up to 2000 items per month. Assuming that the sale statistics will follow a logistic model, identify the specific ODE that will describe the sales of this product and use it to predict how many items will be sold per month by the end of the second year of distribution.

2. The company for which you work has developed a new product and has marketed it for the last two months. In January it sold 20 items, while 25 were sold in February. A market analysis shows that the North-American market could sell up to 2000 items per month. Assuming that the sale statistics will follow a logistic model, identify the specific ODE that will describe the sales of this product and use it to predict how many items will be sold per month by the end of the second year of distribution.

3. A population follows the logistic model, with an initial size of 1000. After 3 years the population is 2500 and its carrying capacity is 10,000. Determine the function that describes the size of this population in terms of time.

4. The population of Canada was approximately 25 millions in 1981, while in 2001 it was approximately 30 millions. If it is growing according to the doomsday model with *c*=0.01, by what date will the universe be full of Canadians?

5. In a famous misquote, since apparently it was never uttered, an IBM CEO apparently said in the 1950's that he saw a world market of at most five computers. Let us say that 5 years ago there were 20,000 computers in Red Deer, that this year there are 100,000, and that the number of computers in Red Deer grows according to a logistic model with a carrying capacity of 1 million.

How many computers will be used in Red Deer in 10 years? The numbers here are as fictitious as the above quote ☺

6. A tank contains 500*L* of water with 5*kg* of salt dissolved in it. Brine consisting of 0.2 *kg*/10*L* of salt is continuously added to the tank at the rate of *10L/min* and instantly mixed. To keep the tank from overflowing, solution is drained from the bottom at the same rate.
   a) How much salt is in the tank at any time during this process?
   b) What amount of salt will be in the tank eventually?

7. One phase in the process of preparation of a certain chemical product consists of injecting a solution consisting of 10% acid (and 90% water) into a sealed tank containing 100 litres of pure water at the beginning. To keep the tank from exploding, the mixture is kept homogeneous by a mixing mechanism and drained at the same rate at which it is poured in, which is 3 litres per minute. Construct the differential equation that describes this process and obtain an explicit solution for it.

8. A certain population starts with 12000 units and grows approximately according to the model $\frac{dP}{dt} = \frac{P}{20}\sin\left(\frac{\pi t}{6}\right)$, where P is the size of the population and t is in months, starting in January. Determine the formula for the function $P(t)$.

9. A certain population starts with 1000 units and grows approximately according to the model $\frac{dP}{dt} = -\frac{P}{30}\cos\left(\frac{\pi t}{6}\right)$, where P is the size of the population and t is in months, starting in January. Determine the formula for the function $P(t)$.

What questions do you have for your instructor?