Chapter 4: Definite integrals and the FTC — *Section 3*

Special types of Riemann sums

What you need to know already:	*What you can learn here:*
➤ What a Riemann sum is.	➤ The key types of Riemann sum that are preferentially used.

In the construction of a Riemann sum, we have to choose which value of *x* to pick to compute the height of each approximating rectangle. It turns out that in the main application that will develop, such a choice does not matter. However, in other applications it may be an issue and all we need to do in this section is to assign names to certain special choices that are often used and have particular features.

We start from two related choices that are made based on the position of the value in each interval.

Knot on your finger:

One-sided Riemann sums

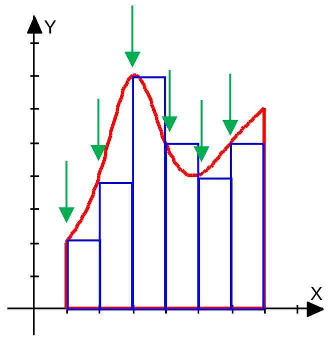
To ***estimate*** the area under a continuous function $y = f(x)$ on the interval $[a, b]$, we can use:

- ***the left Riemann sum*** by picking the left end value of *x* for each slice
- the ***right Riemann sum*** by picking the right end value of *x* for each slice.

Example:

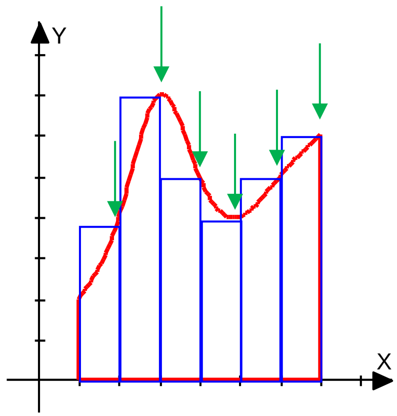
If we want to construct a Riemann sum over the interval $[1, 7]$ based on 6 slices, we are dividing the interval into 6 parts defined by the smaller intervals $[1, 2],[2, 3],[3, 4],[4, 5],[5, 6],[6, 7]$. Therefore:

- for the left Riemann sum we pick all the left end values, thus getting:



$$\frac{3}{6}\left(f(1)+ f(2)+ f(3)+ f(4)+ f(5)+ f(6)\right)$$

- for the right Riemann sum we pick all the right end values, thus getting:



$$\frac{3}{6}\left(f(2)+f(3)+f(4)+f(5)+f(6)+f(7)\right)$$

Both of these choices seem one sided (they are!), so an obvious alternative is to stay in the middle.

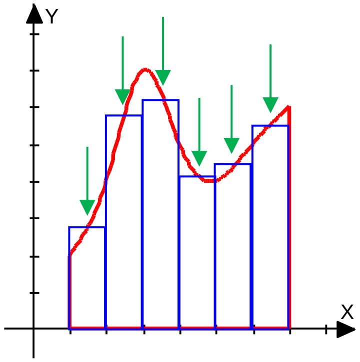
Knot on your finger

Midpoint Riemann sum

To ***estimate*** the area under a continuous function $y=f(x)$ on the interval $[a,b]$, we can use the ***midpoint Riemann sum*** by picking the middle value of *x* for each slice.

Example:

With the same setting as before, the midpoint Riemann sum is given by:



$$\frac{3}{6}\left(f(1.5)+f(2.5)+f(3.5)+f(4.5)+f(5.5)+f(6.5)\right)$$

The last two popular choices are directed by the function itself.

Knot on your finger

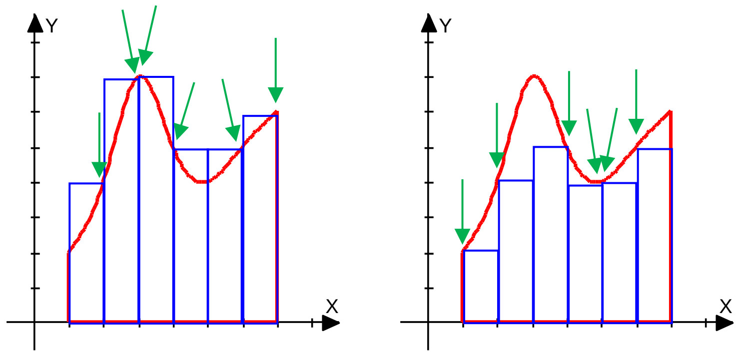
Extreme Riemann sums

To ***estimate*** the area under a continuous function $y=f(x)$ on the interval $[a,b]$, we can use :

- ***the upper Riemann sum*** by picking the largest value of $f(x)$ for each slice
- ***the lower Riemann sum*** by picking the smallest value of $f(x)$ for each slice

Example:

With the same setting as before, here is what the upper (on the left) and lower (on the right) Riemann sums would provide:



All of these choices can be useful at some points, but it is important to keep the following properties in mind.

Quick portrait of

Special Riemann sums

The ***left*** Riemann sum:

- is easy to set up by hand, since the values of x are easy to identify
- is an overestimate if $f(x)$ is decreasing
- is an underestimate if $f(x)$ is increasing

The ***right*** Riemann sum:

- is easy to set up by hand, since the values of x are easy to identify
- is an overestimate if $f(x)$ is increasing
- is an underestimate if $f(x)$ is decreasing

The midpoint Riemann sum:

- is easy to set up by hand, since the values of x are easy to identify
- is decent estimate in most cases of interest.

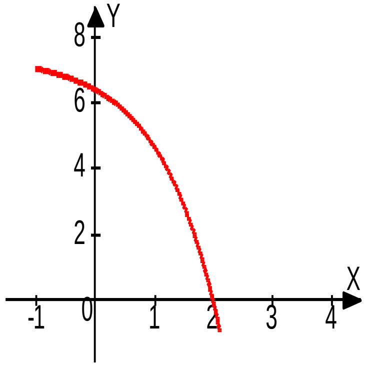
The ***upper*** and ***lower*** Riemann sum:

- are not easy to set up by hand, since the maximum or minimum must be computed separately for each slice
- are poor approximations, since the upper is always an overestimate and the lower is always an underestimate, but...
- provide bounds for the possible values of the correct area.

Can you give me a numerical example?

But of course…

Example:



This is the region bounded by the function $y = e^2 - e^x$ in the first quadrant. To estimate its area by using 4 rectangles, we can use the left sum, which is also the upper sum, since the function is decreasing. In that case we get:

$$x_i = 0, 0.5, 1, 1.5$$

$$A \approx \frac{1}{2}\left[\left(e^2 - e^0\right) + \left(e^2 - e^{0.5}\right) + \left(e^2 - e^1\right) + \left(e^2 - e^{1.5}\right)\right]$$

$$\approx \frac{1}{2}\left[6.39 + 5.74 + 4.57 + 2.91\right] = 9.805$$

We can, alternatively, use a right sum, which is also a lower sum:

$$x_i = 0.5, 1, 1.5, 2$$

$$A \approx \frac{1}{2}\left[\left(e^2 - e^{0.5}\right) + \left(e^2 - e^1\right) + \left(e^2 - e^{1.5}\right) + \left(e^2 - e^2\right)\right]$$

$$\approx \frac{1}{2}\left[5.74 + 4.57 + 2.91 + 0\right] = 6.61$$

So, we know that the area has to be between 6.61 and 9.805.

By using a midpoint sum we get:

$$x_i = 0.25, 0.75, 1.25, 1.75$$

$$A \approx \frac{1}{2}\left[\left(e^2 - e^{0.25}\right) + \left(e^2 - e^{0.75}\right) + \left(e^2 - e^{1.25}\right) + \left(e^2 - e^{1.75}\right)\right]$$

$$\approx \frac{1}{2}\left[6.1 + 5.27 + 3.9 + 1.63\right] = 8.45$$

This value is probably fairly close to the exact value. By using the methods that you will learn in the next sections, we can actually arrive at a much more accurate estimate of 8.389.

Not to contradict what you say, but I got slightly different numbers from those calculations.

Assuming that you have pushed all the right buttons, most likely you used more or fewer decimal digits, thus getting different values. That is fine, since we are just computing approximations. In the next section you will see how to get away from this problem (to a large extent) and arrive at an exact computation of these areas, at least theoretically.

Summary

- Some special types of Riemann sums are used more often than others, as they use particularly convenient choices of representative values.

Common errors to avoid

- Do not confuse left and right sums with upper and lower sums: they are occasionally the same, but not always.

Learning questions for Section I 4-3

Review questions:

1. Explain why it is useful to consider special types of Riemann sums.

2. Describe each special type of Riemann sum, together with its basic properties.

Memory questions:

1. Which values are chosen for a left Riemann sum?

2. Which values are chosen for a right Riemann sum?

3. Which values are chosen for a midpoint Riemann sum?

4. Which values are chosen for an upper Riemann sum?

5. Which values are chosen for a lower Riemann sum?

Computation questions:

For each of the functions provided in questions 1-5, construct the left, right, midpoint, upper and lower Riemann sums to approximate the area bounded by the function and the x-axis between the two given values and with the given number of intervals.

1. $y = x^2$ on $[0, 4]$, n=4.

2. $y = \sin x$ on $[0, \pi]$ n= 6.

3. $y = \ln x$ on $[1, 6]$, n= 5.

4. $y = \cosh^2 x$ on $[1, 2.5]$, n=6.

5. $y = e^{-x^2}$ on $[0, 5]$, n=5.

Application questions:

1. If the velocity of a moving object is given by the function $v = t^2 - 12t + 36$, where t is in seconds and v in m/sec, estimate the distance travelled by the object in the first 6 seconds by using 6 rectangles and the midpoint sum.

Templated questions:

1. Choose a function that is positive on an interval of your choice and construct the special Riemann sums that approximate the area of the region it bounds.

What questions do you have for your instructor?