Chapter 4: Definite integrals and the FTC — **Section 6**

The Fundamental Theorem of Calculus:

The theoretical version

What you need to know already:	What you can learn here:
➤ Definite Riemann integrals and their basic properties ➤ Derivatives and antiderivatives.	➤ The key tool to solve the area problem, namely the area function, and its key property.

The Riemann integral provides a *theoretical* method for defining the area under a curve, but it still has a major weakness: it does not provide a simple way to *compute* the integral it defines. Attempting to compute the limit that provides such definition is a nightmare in the simplest cases and simply out of the question for less than trivial examples. If no easier method had been developed to compute a definite integral, calculus would likely still be an obscure and marginal area of mathematics and few of its application would have seen the light of day.

But calculus is widely known and used because of a computational method known, deservedly, as the *Fundamental Theorem of Calculus* (FTC). Most of the facts stated in the theorem were actually discovered by Newton's teacher, Isaac Barrow, but his argument was very geometrical, not algebraic, so he did not fully realize what he had accomplished. But under the genius of Newton and Leibniz it was developed into an organized set of methods that opened the doors to modern technology.

In this note I will present the smart idea behind the method, while the next sections will show you how to use this idea to arrive at the practical implementations of the FTC.

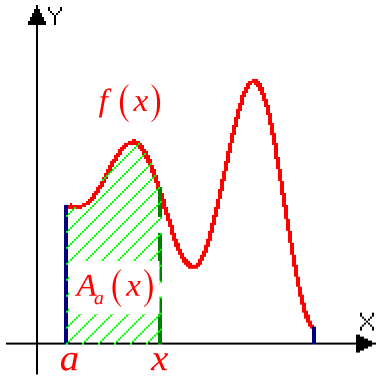
Definition

Consider a function $y = f(x)$ that is continuous on an interval containing the number a. Its ***area function*** $A_a(x)$ is defined as:

$$A_a(x) = \int_a^x f(t)dt$$

Notice that in the above notation, the capital A refers to the fact that the function represents an area, while the lower case a of the subscript refers to the fact that the region starts at the value a.

This function is depicted graphically by the following picture.



Notice that the x of the area function refers to the *variable* that appears as the right limit of the region, while the t is a *different copy* of the same variable, used to define the integral. This may feel confusing at the beginning, but, hopefully, you will get used to the double use of the horizontal coordinate in this setting.

Notice also that this area function is exactly what we need to know if we want to be able to compute definite integrals, so the more we know about it, the better. In particular, computing its derivative turns out to be an essential piece of information, and that is easier than you may think.

Technical fact
Fundamental Theorem of Calculus:
Theoretical version of the FTC

Given a continuous function $f(x)$ and any value a in its domain, the ***derivative of the area function*** $A_a(x)$ ***is the original function*** itself:

$$\frac{d}{dx}A_a(x)=\frac{d}{dx}\left(\int_a^x f(t)dt\right)=f(x)$$

Proof

As usual, I will just give you a sketch of the proof and avoid the technical details. You can check the latter in any standard textbook

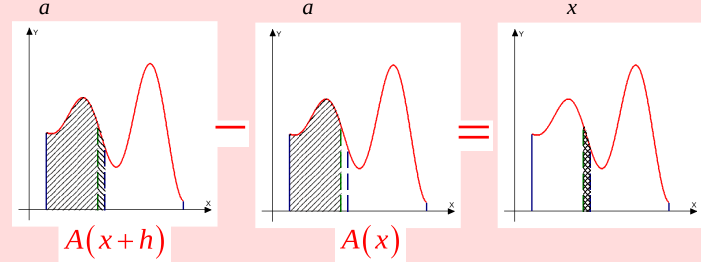
By the definition of derivative:

$$\frac{d}{dx}A_a(x)=\lim_{h\to 0}\frac{A_a(x+h)-A_a(x)}{h}$$

$$=\lim_{h\to 0}\frac{\int_a^{x+h} f(t)dt-\int_a^x f(t)dt}{h}$$

By using the additive property of definite integrals, presented here in terms of graphs, we notice that the numerator of this fraction is the area of a thin slice from x to $x+h$.:

$$\int_a^{x+h} f(t)dt \quad - \quad \int_a^x f(t)dt \quad = \quad \int_x^{x+h} f(t)dt$$



But we know that:

$$\int_x^{x+h} f(t)dt \approx f(x)h$$

Moreover, this approximation becomes better as h becomes smaller, which is what we are doing with the limit. Therefore we can conclude that:

$$\frac{d}{dx}A(x)=\lim_{h\to 0}\frac{f(x)h}{h}=f(x)$$

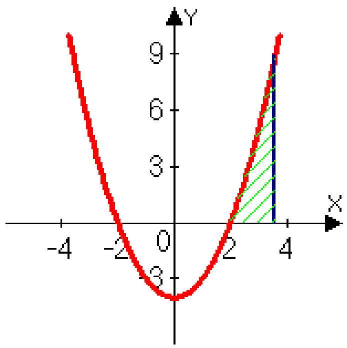
Notice that the theorem, as stated above, does not require $f(x)$ to be positive, even though I assumed it in my sketch of the proof. However, when $f(x)$ is not positive, the interpretation of the area function has to be done in terms of positive and negative areas as we discussed in relation to the definition of Riemann integrals.

Example: $y = x^2 - 4$

This function is continuous for every real number, so we can consider, for instance, the area function:

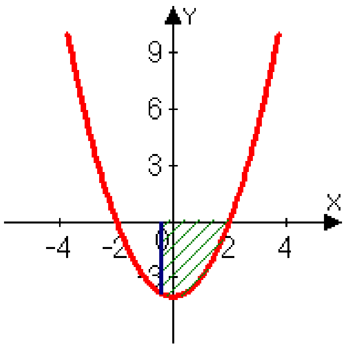
$$A_2(x) = \int_2^x (t^2 - 4)\, dt$$

For $x > 2$, this represents the area under the curve from 2 to *x*, by the definition of definite integral.

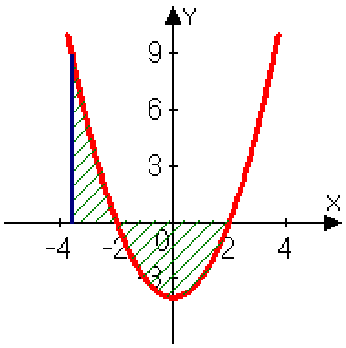


For $-2 < x < 2$, this still represents the area under the curve from 2 to *x*, since in this case:

$$A_2(x) = \int_2^x (t^2 - 4)\, dt = -\int_x^2 (t^2 - 4)\, dt = -(-A) = A$$



However, for $x < -2$ the area function must be interpreted as a combination of area values. I leave to you the task to figure out which ones and how are they combined, while offering the related pictures.



The theoretical version of the FTC tells us that the derivative of this area function is given by:

$$\frac{d}{dx} A_2(x) = \frac{d}{dx}\left(\int_2^x (t^2 - 4)\, dt \right) = x^2 - 4$$

In order to make sure that you are clear on the meaning of this version of the FTC, a version that is probably unfamiliar to you, it may be useful to work through a question like the one of the following example.

Example: $\dfrac{d}{dx}\displaystyle\int_0^{\sin x} e^{t^2}\,dt$

To compute this expression, we notice that it is the derivative of a definite integral, so it looks like the derivative of an area function, to which we could apply the FTC. However, the top limit is not just *x*, as required by the theorem, but a function of it, so we need to use the chain rule. If we let $u=\sin x$ we have:

$$\frac{d}{dx}\int_0^{\sin x} e^{t^2}\,dt=\frac{d}{dx}\int_0^{u} e^{t^2}\,dt=\left(\frac{d}{du}\int_0^{u} e^{t^2}\,dt\right)\times\frac{du}{dx}$$
$$=e^{u^2}\cos x=e^{\sin^2 x}\cos x$$

If this does not look very exciting or useful, remember that this is only the theoretical version. Once you see the practical version, which will give concreteness and remind you of familiar facts from high school, a smile should appear on your face ☺.

Summary

- The area function for a continuous function $f(x)$ is differentiable, and its derivative is $f(x)$.

Common errors to avoid

- Don't get confused in the notation of the FTC; instead make sure to understand what each detail of such notation represents and what the statement of the theorem itself means.

Learning questions for Section I 4-6

Review questions:

1. Explain what the theoretical version of the FTC states.

Memory questions:

1. Which formula identifies the theoretical version of the FTC?

2. Which condition is required for the integrand in order for the FTC to be valid?

Computation questions:

Compute the derivatives requested in each of questions 1-8.

1. $\frac{d}{dx}\int_0^x \frac{e^{-\sin t}}{\cosh t}\,dt$.

2. $\frac{d}{dz}\int_1^z \left(3x-\frac{1}{\sqrt{x}}\right)dx$

3. $\frac{d}{dx}\int_x^6 e^{t^2}\,dt$.

4. $\frac{d}{dt}\int_1^{\ln t}\left(\frac{1}{x}+\sqrt{x}\right)dx$

5. $\frac{d}{dx}\int_0^1 \left(3\sin x+e^x\right)dx$

6. $\frac{d}{dx}\int_{\ln x}^{2.5} \cosh^2 t\,dt$

7. $\frac{d}{dx}\left(\int_{-x^2}^{0} \cos^2(t)\,dt\right)$.

8. $\frac{d}{dx}\left(\int_{x^2+1}^{0} \cosh^2(t)\,dt\right)$

9. Perform a first derivative analysis on the function $f(x)=\int_0^x (1-2t)e^{-t^2}\,dt$.

Theory questions:

1. In order to apply the theoretical version of the FTC, what do we need as limits of integration?

2. Does the FTC state that $\frac{d}{dx}\int f(x)dx = f(x)$?

3. Does the FTC state that the derivative of an antiderivative is the original function?

4. Is it true that $\frac{d}{dx}\int_1^{2.5} f(t)dt = f(x)$?

5. What is the derivative of a function of the form $B(x) = \int_a^{g(x)} f(t)\,dt$?

Templated questions:

1. Choose an interval $[a, b]$ and function $f(x)$ that is continuous over it, construct the area function $A_a(x)$ for it and determine the derivative of such area function.

What questions do you have for your instructor?