Chapter 4: Definite integrals and the FTC — *Section 7*

The Fundamental Theorem of Calculus:

The practical version

What you need to know already:	*What you can learn here:*
➤ The theoretical version of the FTC.	➤ The practical version of the FTC. No kidding!

The theoretical version of the FTC gives us a nice piece of information about the area function and the definite integral that defines it, but still does not tell us how to compute either. Or, to say better, does not tell us explicitly. To do that, we only need to turn its statement around a bit.

Technical fact
Fundamental Theorem of Calculus:
Practical version of the FTC

Given a function $y = f(x)$ that is ***continuous*** on the interval $[a,b]$, and given any antiderivative $F(x)$ of $f(x)$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof

The theoretical version of the FTC tells us that the derivative of $A_a(x)$ is $f(x)$. But this means that $A_a(x)$ must be an antiderivative of $f(x)$, but which of the infinitely many antiderivatives is it?

To find out, let's pick any one antiderivative, say $F(x)$, and notice that, since we are assuming $f(x)$ to be continuous, $A_a(x)$ must differ from $F(x)$ only by an additive constant: $A(x) = F(x) + c$

But we also know at least one value of $A_a(x)$, since we know that:

$$A_a(a) = \int_a^a f(t)dt = 0$$

Therefore $0 = F(a) + c$, $c = -F(a)$ and $A(x) = F(x) - F(a)$. From this follows that:

$$\int_a^b f(t)dt = A_a(b) = F(b) - F(a)$$

Needless to say, but I'll say it anyway, since we have a choice of which antiderivative to use in the application of the FTC, we choose the one whose constant of integration is $c=0$, that is, we don't bother adding the constant.

I admit that this is cool and I enjoyed the FTC in high school, when we were first told about it. But why is it called Fundamental?

Very legitimate question, since in mathematics we don't call any statement *fundamental*, or *central*, or *important*, but only those that deserve the title. So, here is why.

Knot on your finger

The ***Fundamental Theorem of Calculus*** (FTC) is called *Fundamental* because:

- It allows us to ***solve the area problem*** in its generality, as we shall see soon.
- It solves the area problem in a very ***computationally efficient*** way.
- It shows that two very important ***problems*** in mathematics, ***finding slopes and finding areas***, that seemed unrelated, are in fact ***strictly linked***.
- Since the same idea of definite integral used to solve the area problem ***can be applied to many other geometrical and physical problems***, the FTC allows us to solve many more important problems in the physical sciences.
- Such solutions have allowed, in recent time, the development of the ***high level of technology*** that we enjoy today.

I think that this is exciting: many more doors are now open for us to explore.

Are we going to see those uses soon?

Very soon, but first let us fix a detail in the notation and look some more at how elegantly the FTC solves the area problem.

Knot on your finger
Definite integral notation

If $F(x)$ is the antiderivative of $y=f(x)$ that we use to apply the practical version of the FTC, we write, as an intermediate step aimed at clearly identifying such antiderivative:

$$\int_a^b f(x)dx = \left[F(x)\right]_a^b = F(b)-F(a)$$

Other, similar notations can also be found in the literature, such as:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b)-F(a)$$

Example: $y=\sin^{-1}x,\ y=0,\ \ x=0,\ x=1$

These three curves bound the region in the first quadrant under the given function and to the left of $x=1$. Its area is therefore defined by

$$\int_0^1 \sin^{-1} x dx$$

If we use the definition of integral, we would need to compute:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sin^{-1} x\ \Delta x_i$$

Even if we assume all rectangular slices to have the same length, so that all Δx_i are equal, this is an impossible limit to figure out. But, thanks to the FTC, its computation boils down to computing an antiderivative of $y = \sin^{-1} x$. By using integration by parts you can check that this is:

$$\int \sin^{-1} x dx = x\sin^{-1} x + \sqrt{1-x^2} + c$$

The rest is now easy by using the FTC:

$$\int_0^1 \sin^{-1} x dx = \left[x\sin^{-1} x + \sqrt{1-x^2} \right]_0^1$$

$$= \left(1\frac{\pi}{2} + 0\right) - (0+1) = \frac{\pi}{2} - 1 \approx 0.57$$

The FTC is simple enough that further examples should be unnecessary. The learning questions will provide ample illustrations and experience for you. Moreover, we shall see many applications of this theorem and, with them, many examples of how to use it. If you do need further worked out examples, there are many web sites that provide them. Ain't it great when you are studying something difficult, but well known and used?

And, speaking of applications and uses, here is a first, very applied and very general consequence of the FTC.

Technical fact
The net change theorem

If a quantity $Q(t)$ changes continuously during the time period from *a* to *b*, then the ***net change*** in this quantity during this period is given by:

$$Q(b) - Q(a) = \int_a^b Q'(t)dt$$

Proof

This is just the statement of the practical version of the FTC, but with an equally practical interpretation!

This fact is particularly useful when the function describing the quantity is not known, but its rate of change is, as we saw when studying ODE's.

Example: $v(t) = t^2 - 2t$

If an object is moving on the *y*-axis so that its velocity is given by this function, its net change in the position for the first five seconds is given by:

$$y(5) - y(0) = \int_0^5 \left(t^2 - 2t\right) dt = \left[\frac{t^3}{3} - t^2\right]_0^5 = \frac{50}{3}$$

Notice, however, that this is not the total distance travelled by the object during this time, since the object may have moved back and forth within that time period. In fact, since:

$$t^2 - 2t = t(t-2)$$

we can see that the object travels down from $t = 0$ to $t = 2$, but travels up after $t = 2$. Therefore, as we did for computing a total area, the total distance it travels is:

$$T = -\int_0^2 \left(t^2 - 2t\right) dt + \int_2^5 \left(t^2 - 2t\right) dt =$$

$$= -\left[\frac{t^3}{3} - t^2\right]_0^2 + \left[\frac{t^3}{3} - t^2\right]_2^5 = \frac{58}{3}$$

So, this is the first application of definite integral outside of areas, and as it befits a method that was invented to describe the motion of planets, it relates to moving objects. Many more uses to come, just stay tuned and excited!

And speaking of practical things related to calculus, it seems that one of the factors contributing to Newton's death, if not the main cause, was a calculus problem, not of the kind we are dealing with here, but in the sense of stones (in medical terms, a *calculus*) in his bladder that damaged his digestive system. And, wouldn't you know, Leibniz died while suffering from gout, a disease caused by the accumulation of uric acid crystals (calculus again!) in the joints! Beware!

Summary

- A definite integral whose integrand is continuous may be computed as the difference between the values of one of its antiderivatives at the two endpoints.
- This fact provides a practical solution to a wide variety of area problems and will provide the general solution to this problem, as well as to many other applied problems.
- In particular, the practical version of the FTC allows us to easily compute the net change of a quantity whose rate of change is known.

Common errors to avoid

- The FTC looks and is easy to understand and implement. But don't forget that behind it there is a large amount of algebra and indefinite integration that must be done correctly in order for the theorem to work correctly.

Learning questions for Section I 4-7

Review questions:

1. Explain what the practical version of the FTC states.
2. Describe what the net change theorem allows us to do.

Memory questions:

1. What does the practical version of the FTC state?
2. Which one condition on the integrand is required by the practical version of the FTC?

Computation questions:

In each of questions 1-6, use the practical version of the FTC to evaluate the given definite integral.

1. $\int_1^4 \left(3x - \frac{1}{\sqrt{x}}\right) dx$

2. $\int_0^1 \left(3\sin x + e^x\right) dx$

3. $\int_0^{\pi/4} \frac{3}{1-\sin^2 x}\, dx$

4. $\int_1^2 \frac{\ln x}{x^2}\, dx$

5. $\int_0^{3/5} \frac{x^3}{\sqrt{1-x^2}}\, dx$

6. $\int_0^1 \tan^2 x\, dx$

In each of questions 7-10 compute the area of the region bounded by the given curves.

7. $y = \sin x,\ y = 0,\ x = 0,\ x = \pi$.

8. $y = e^{2x},\ y = 0,\ x = 0,\ x = 1$.

9. $y = \begin{cases} \sin x & if\ -1 \le x \le 0 \\ 2x^2 & if\ 0 \le x \le \pi \end{cases}$ and the x-axis.

10. $y = 1 - e^{2x},\ y = 0,\ x = 0,\ x = 1$

11. Compute the derivative of the function $h(t) = \int_1^t \left(\frac{1}{x} + \sqrt{x}\right) dx$ in the following two ways and verify that they provide the same answer:

   a) evaluating the integral and then differentiating it.

   b) Applying the theoretical version of the FTC.

12. Compute each of the following seemingly similar expressions involving integrals.

   a) $\int \left(3\sin x + e^x\right) dx$

   b) $\int_0^1 \left(3\sin x + e^x\right) dx$

   c) $\frac{d}{dx}\int \left(3\sin x + e^x\right) dx$

   d) $\frac{d}{dx}\int_0^1 \left(3\sin x + e^x\right) dx$

   e) $\frac{d}{dx}\int_0^x \left(3\sin t + e^t\right) dt$

13. Determine the net change of the quantity $Q(x)$, whose derivative is $Q'(x) = xe^{2x}$, between $x = 0$ and $x = 1$. In your answer clearly exhibit all major steps and identify the one where the FTC is used.

14. Determine the net change of the quantity $Q(x)$, whose derivative is $Q'(x) = x\sin 2x$, between $x = 0$ and $x = 1$. In your answer clearly exhibit all major steps and identify the one where the FTC is used.

Theory questions:

1. Can we use the practical version of the FTC to compute $\int_{-1}^{1} x\ln^2 x dx$
2. Use the formula of the FTC to compute the area of the region bounded by the curves $y = \frac{3}{x-1}$, $y = 0$, $x = 0$, $x = 2$. How do you explain the conclusions of your work?
3. What is the relation between indefinite and definite integrals?
4. State two reasons why the FTC is called *fundamental*.

Proof questions:

1. Use the FTC to determine the value of $\int_{1}^{-1} (x^3 - x) dx$. How do you explain the answer you get, assuming you get the correct one?
2. Prove that $\int_{0}^{\pi/2} \sin^2 x dx = \frac{\pi}{4}$ by using the FTC and also, without using the FTC, but instead using the graphical meaning of the identities $\sin^2 x + \cos^2 x = 1$ and $\sin x = \cos\left(\frac{\pi}{2} - x\right)$.

Application questions:

1. If $m' = f(v)$ represents the rate at which the mass of an object changes as its speed *v* changes (according to the theory of relativity), what does $\int_{a}^{b} f(v) dv$ represent? In the *cgs* system, what are units of $f(v)$?

What questions do you have for your instructor?