The four steps process for building integrals

What you need to know already:

- How to compute areas by using integrals.

What you can learn here:

- How to solve a problem through integrals by building up a Riemann sum formula.

This section is a bit unusual, in that it is a background reflection on a method that will become clear only through some concrete applications. Yet, this reflection is an important one, since it will clarify the mechanism behind all such later applications. So, when you first read the following, spend some time just going over it and be patient, as there will not be any examples or exercises! Then review this section as you go through the applications that use this method, so as to connect theory and practice in an effective way.

In the previous chapter, we developed the definite integral in order to find a general method for computing areas of regions with curved boundaries. But we then noticed that the Riemann formula we constructed to solve such problem is an algebraic formula that makes sense independently from the original problem of computing areas. In fact, one of the first difficulties we encountered when dealing with definite integrals was how to connect them properly to areas: some integrals represent areas, others represent differences of areas; some areas are represented by a single integrals, some by the sum and difference of several integrals.

It turns out that this apparent difficulty is in fact an advantage. The algebraic nature of the Riemann integral formula allows us to use it in situations which have nothing to do with areas. To be able to use it in such situations, let us remember the main steps we go through to construct a Riemann integral:

1) We divide the region into several thin vertical slices.
2) We approximate the area of each slice by using the simple formula for the area of a rectangle.
3) We add up the areas of the rectangles so obtained, thus constructing a Riemann sum.
4) We take the limit of such sum as the number of slices goes to infinity and the thickness of each goes to 0, thus getting the Riemann integral.

Once all this is done, we can then use the FTC to actually compute the horrendous limit formula generated by the process.

Well, scholars realized very quickly that this process can be applied to other quantities that are not areas. All we have to do is generalize the description so as not to be stuck with areas only. This leads to a four step process that will prove very valuable in many applications to follow.

So, let us assume that we have to compute a quantity $Q$ such that:

- $Q$ can be computed through a simple formula in a simple situation.
- The situation we need to address is more complicated than the one for which the simple formula for $Q$ can be used (stay with me!).
- But we can slice our problem into smaller pieces (remember the salami technique!) in such a way that the value of each piece is well approximated by the simple formula.

This is similar to a cartoon, where a whole movement is divided into individual frames, each depicting a simple image.

In such a situation, integrals become the magic bullet, since their construction works in the same way. This provides the idea for a general strategy.
The Four Step Strategy for computing a quantity through a definite integral

To compute a total quantity $Q$ whose size depends on a variable $x$ for $a \leq x \leq b$:

- **Slice** the whole range of values of $x$ into $n$ small intervals, each of width $\Delta x_i$, preferably with $\Delta x = \frac{b-a}{n}$ for all slices, so that each slice gives rise to a portion $Q_i$ of the quantity.

- **Approximate** such portion, that is, use a simple formula for $Q_i$ which involves $\Delta x$ through a formula of the form $Q_i = f(x_i)\Delta x$.

- **Add up** the values of all such approximations, thus getting a Riemann sum of the form:

$$Q \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

- **Take the limit** of this approximation as the number of slices approaches infinity and their width approaches 0, thus obtaining the Riemann integral:

$$Q = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_{a}^{b} f(x)dx$$

For obvious reasons, I will often refer to this standard four-step process as the **slice-approximate-add-limit** process. Well guess what: it works like a charm, despite its initial appearance of complexity! Just stay tuned to the next sections to see how.

On a historic note, this method is not new, as it was known to and used by Eudoxus of Cnidus, a Greek scholar of the fourth century B.C. and was also used by Archimedes for some of his volume and area calculations. What they did not know, was how to use integrals (not developed until much later) for the numerical evaluation of the quantity in question **in general**, so that their method only worked when the limit of the last step (again a much later concept) became apparent through some ad hoc simplification.

It is most likely that Newton and Leibniz relied on the work of these ancient scholars – and others – to develop their calculus method. Which explains Newton’s famous quote: “If I have seen further it is by standing on the shoulders of Giants.”

So, do not hesitate to stand on similar shoulders, but always academically speaking!