

## Volumes by slicing

### What you need to know already:

- ▶ The four step process to set up an integral.
- ▶ Basic geometric formulae for areas and volume

### What you can learn here:

- ▶ How to set up the integral representing the volume of a solid for which we know the areas of the cross sections.

The four-step process of *slice-approximate-add-limit* can also be used to compute the volumes of solids for which we have some basic information. This application of the method is as basic as sliced bread. Literally!

Think of a loaf of bread and imagine you have to compute its volume.



The good old trick of immersing it in water and measuring the amount of displaced water does not work here for obvious reasons, but we can use integrals instead, as follows.

- 1) Imagine the loaf laid along the  $x$  axis and sliced into slices of equal thickness  $\Delta x$
- 2) Assume you can compute the area of each slice in position  $x$  and label that as  $A(x)$ . This can be done by using basic geometric formulae or by applying the integral method itself. This is not always easy or feasible, but always theoretically possible.
- 3) The volume of each slice can now be approximated by multiplying the area  $A(x)$  by the thickness  $\Delta x$ .

- 4) Adding up the volumes of all slices gives an approximation of the volume of the whole loaf.
- 5) Taking the limit as the thickness goes to 0 provides the exact volume of the loaf.

In calculus terms, we do the following:

- ▶ **Slice:** Compute the thickness  $\Delta x = \frac{b-a}{n}$
- ▶ **Approximate:** Compute the approximate area of each slice  $A(x)$  and use it to compute the approximate volume of each slice:  $\Delta V \approx A(x)\Delta x$
- ▶ **Add up:** Use the sigma notation to obtain an approximation of the whole volume:  $V \approx \sum_{i=1}^n A(x_i)\Delta x$
- ▶ **Take the limit:** Obtain the exact volume by letting the number of slices approach  $\infty$  and their thickness go to 0:  $V = \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i)\Delta x$ .

The last expression describes a Riemann integral, and this gives us the general strategy.

*Strategy for using integrals as  
volumes of solids of  
known cross sectional area*

The **volume** of a solid whose cross sectional area at position  $x$ , with  $a \leq x \leq b$ , is  $A(x)$  is given by:

$$V = \int_a^b A(x) dx$$

Notice that this method relies heavily on our ability to obtain the function  $A(x)$ . We shall see through examples how that can be done in special cases (the ones we expect you to handle). For more complex situations you may need to wait until more advanced methods become part of your knowledge.

*Example:*



Let us compute the volume of a square pyramid whose base has side  $s$  and whose height is  $h$ .

How do we visualize this? We start by sketching a picture and identifying the direction we want to use as the  $x$  axis.

Any cross section at height  $x$  is a square whose side  $w$  depends on  $x$ , according to the side view shown here.

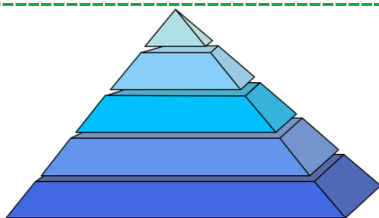
Therefore:

$$\frac{s}{h} = \frac{w}{h-x} \Rightarrow w = \frac{s(h-x)}{h}$$

This means that at height  $x$ , the area of the cross section is:

$$A = w^2 = \frac{s^2(h-x)^2}{h^2}$$

The slicing we are using corresponds to this image, although the image uses slant sides and thick slices, while the integral's slices are thin approximations with vertical sides, with the approximation going away in the limit.



Since  $s$  and  $h$  are constant, it follows that the volume of the pyramid is:

$$V = \frac{s^2}{h^2} \int_0^h (h-x)^2 dx = \frac{s^2}{h^2} \left[ -\frac{(h-x)^3}{3} \right]_0^h = \frac{s^2}{h^2} \left[ \frac{h^3}{3} \right] = \frac{s^2 h}{3}$$

This is indeed the standard formula for the volume of such a pyramid.

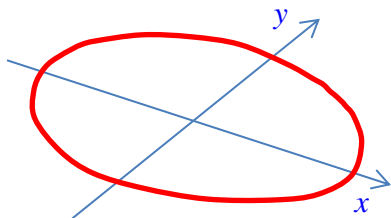
And now, let's look at a less familiar solid.

**Example:**

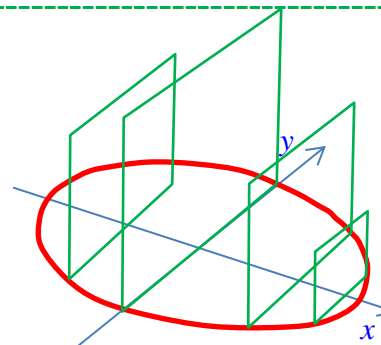
*Compute the volume of the solid whose base is a circular disk of radius  $r$  and whose cross sections perpendicular to the  $x$  axis are squares.*

How do we visualize this? With great difficulty, especially when trying to sketch a 3D object on the two dimensions of the page.

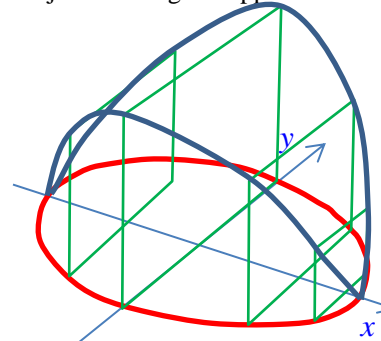
We start by noticing that the equation of the base is  $x^2 + y^2 = r^2$ , so we sketch that in perspective.



Next we draw squares perpendicular to the  $x$  axis as sections.



And now imagine them joined along the upper vertices



Ok, use your imagination a bit, eh? ☺

Now back to the computational part. Each square constructed at level  $x$  will have a base long  $2|y|$ . But:

$$x^2 + y^2 = r^2 \Rightarrow |y| = \sqrt{r^2 - x^2}$$

Therefore, the area of each square will be:

$$A = (2|y|)^2 = 4(r^2 - x^2)$$

The values of  $x$  range from  $-r$  to  $r$ , so that the required volume is given by:

$$\begin{aligned} V &= 4 \int_{-r}^r (r^2 - x^2) dx = 4 \left[ r^2 x - \frac{x^3}{3} \right]_{-r}^r \\ &= 4 \left[ \left( r^3 - \frac{r^3}{3} \right) - \left( -r^3 + \frac{r^3}{3} \right) \right] = \frac{16}{3} r^3 \end{aligned}$$

*I must say that the calculus part is more understandable than the visual part!*

That is because at this point you are more familiar with the calculus part than with the visual description of the solids we are playing with. But there is a cure for

that and, as usual, it is practice. So, off you go, experimenting with some similar problems. But not too many, since in the next three sections we'll see a particularly simple case of solids whose sections are easy to draw and compute. There is light at the end of the solid!

### *Summary*

- When the area of each cross section of a solid along an axis is known, it is possible to set up an integral that provides the volume of such solid.

### *Common errors to avoid*

- Visualizing a solid described in terms of its cross sections can be difficult and can lead to setting up the integral incorrectly.

### *Learning questions for Section I 5-5*

#### *Review questions:*

1. Describe how the formula for the volume of a solid is constructed by slicing.
2. Describe how the four steps in the construction of an integral are implemented for the volume of a solid.
3. Identify the geometric meaning of each portion of a definite integral that represents the volume of a solid obtained by slicing.

#### *Memory questions:*

1. What is the formula for the volume of a solid region whose cross-sectional area between  $a$  and  $b$  is given by  $A(x)$ ?

### Computation questions:

For each of the solids described in questions 1-4, construct the integral that determines its volume and, if possible, compute it.

1. The solid whose base is a semicircle of radius  $r$  and whose sections perpendicular to the diameter of the semicircle are isosceles right triangles with the right angle vertex on such diameter.
2. The solid whose base extends from  $x=0$  to  $x=5$ , between the  $x$  axis and the parabola  $y = x^2 + 1$  and whose sections perpendicular to the  $x$ -axis are squares.
3. The solid whose base is the triangle at  $(0, 0)$ ,  $(3, 0)$ ,  $(0, 2)$  and whose cross section perpendicular to the  $y$ -axis are semicircles.
4. The solid whose base is the triangle at  $(0, 0)$ ,  $(2, 0)$ ,  $(0, 3)$  and whose cross section perpendicular to the  $y$ -axis are rectangles with a height that is double the width.

### Templated questions:

1. Devise a solid whose sectional area is possible to compute, try to visualize it and then set up the integral that describes its volume.

***What questions do you have for your instructor?***

