Roberto’s Notes on Integral Calculus

Chapter 5: Basic applications of integration

Section 6

Volumes by disks and washers

What you need to know already:

➢ The four-step process to construct an integral.
➢ The method of computing volumes by slicing.

What you can learn here:

➢ How to set up integrals describing volumes of solids of revolution.

We have seen that the four-step process of slice-approximate-add-limit can be used very effectively to compute volumes of solids whose cross sections have a computable area. In particular, this applies to solid of revolution, since in that case their sections are either circles or rings. Of course, this is the same idea we saw for surfaces of revolution, except that this time we rotate a region, instead of a curve, around an axis, thus obtaining a solid, instead of a surface. And, of course, we now want to find a volume, rather than a surface area. Just as a reminder, here are three familiar objects that are solids of revolution, shown together with the axis of rotation and the region whose rotation generates the solid.

Notice that for the bowling pin and the capsule, the axis makes up one part of the boundary of the region. When we rotate such region around the axis, each section is a circle whose area, as we all know, is given by \( A = \pi r^2 \).

If we know the size of the radius in terms of the position of the object along the axis of rotation, we can apply the method of slicing and arrive at the method of computing volumes by disks.

Why do we call it by “disks” rather than by “circles”?

As you remember from the last section, when we multiply the area of a cross-section by \( \Delta x \) we obtain the volume of a slice. Well, a slice in this case looks like a disk, a hockey puck, if you prefer.

\[
V_i = \pi r_i^2 \Delta x
\]
Since this terminology was NOT invented by Canadians, the name “disks” stuck.

**Strategy for computing volumes by disks**

The volume of a **solid of revolution** whose sectional area at level \( x \) along the axis of rotation is a **disk** of radius \( r(x) \), for \( a \leq x \leq b \) is given by the integral:

\[
V = \int_{a}^{b} \pi r(x)^2 \, dx
\]

**Example:** \( f(x) = \sqrt{2 - (x - 1)^2}, \; g(x) = 1, \; 0 \leq x \leq 1 \)

If we rotate this region (left) around its bottom boundary \( y = 1 \) we obtain the bullet-shaped solid shown on the right.

Each vertical slice of the region, when rotated becomes a disk whose radius is given by the difference between circle and line. Therefore, we can conclude that the volume of such bullet is given by:

\[
V_i = \pi R^2 \Delta x - \pi r_i^2 \Delta x = \pi \left( R^2 - r_i^2 \right) \Delta x
\]

We know how to compute the volumes of both cylinders. Therefore, the volume of each slice is:

\[
V_{i} = \pi R_x^2 \Delta x - \pi r_i^2 \Delta x = \pi \left( R^2 - r_i^2 \right) \Delta x
\]

This integral is not immediate, but can be computed: try it as an exercise.

If we rotate the region around an axis that is not part of its boundary, the resulting solid will have a hole, as in the case of the beer bottle. In fact each slice will have a hole and will look like a flat ring, or as a CD, if the hole is small.

As before, what we really need is the volume of such slice, which will look like a cylinder with a cylindrical hole in the middle.

So, if we can figure out how to express the bigger and smaller radii in terms of the variable \( x \) we are using, we can set up the needed integral.
**Strategy for computing volumes by washers**

The volume of a *solid of revolution* whose sectional area at level \( x \) along the axis of rotation is a *washer* of radii \( R(x) \geq r(x) \), for \( a \leq x \leq b \) is given by the integral:

\[
V = \int_a^b \pi \left( R^2 - r^2 \right) dx
\]

*Don’t tell me: we call it the “washer” method because it was invented before CD’s, eh?*

It certainly isn’t because they have something to do with hygiene! Let’s look at some examples.

**Example:** \( f(x) = \sqrt{2 - (x - 1)^2}, g(x) = 1, 0 \leq x \leq 1 \)

This time we rotate this same region around the \( x \) axis, thus getting a solid looking like some kind of sleeve.

Each vertical slice of the region, when rotated, becomes a washer, with outer radius given by the square root function (in fact, a portion of a circle) and inner radius given by the line. Therefore, we can conclude that the volume of such sleeve is given by:

\[
V = \int_0^1 \pi \left[ (\sqrt{2 - (x-1)^2})^2 - (1)^2 \right] dx = \pi \int_0^1 \left[ 2 - (x-1)^2 - 1 \right] dx
\]

Computing the value of this integral is now routine.

**Example:** \( f(x) = \sqrt{2 - (x-1)^2}, g(x) = 1, 0 \leq x \leq 1 \)

Same region again, but this time rotated around the line \( y = 2 \).

We again get a sleeve, with each slice being a washer. The outer radius is the distance from the bottom boundary to the axis, which is 1, while the inner radius is the distance between circle and axis of rotation. Therefore, we can conclude that the volume of such sleeve is given by:

\[
V = \int_0^1 \pi \left[ 1^2 - \left( \sqrt{2 - (x-1)^2} \right)^2 \right] dx
\]

Again, nothing unusual with the computation, but watch for the algebraic details!
**Example:** \( f(x) = \sqrt{2-(x-1)^2} \), \( g(x) = 1, \, 0 \leq x \leq 1 \)

Finally, if we rotate the same region around the \( y \) axis, we get this strange looking object.

Notice that this time to see the washers we need to slice horizontally, so that the variable to consider is not \( x \), but \( y \). With this in mind, the outer radius is the right boundary, that is 1, while the inner radius is \( x \) coordinate of the circle:

\[
y = \sqrt{2-(x-1)^2} \quad \Rightarrow \quad y^2 = 2-(x-1)^2 \quad \Rightarrow \quad (x-1)^2 = 2 - y^2
\]

\[
\Rightarrow \quad x-1 = -\sqrt{2 - y^2} \quad \Rightarrow \quad x = 1 - \sqrt{2 - y^2}
\]

Notice that we need to choose the negative root, since \( x \leq 1 \) in our region.

Moreover, since we want \( 0 \leq x \leq 1 \), we need \( 1 \leq y \leq \sqrt{2} \). Therefore, we can conclude that the volume of such solid is given by:

\[
V = \pi \int_1^{\sqrt{2}} \left[ 1^2 - (1 - \sqrt{2 - y^2})^2 \right] dy
\]

Notice that in this case the integral can be constructed and even computed, but the set up in terms of \( y \) generates some difficulties that we wish we could avoid. Well, take heart: we’ll do that in the next section, by using a different kind of slicing.

**Summary**

- With solids of revolution, the slicing method provides a fairly simple and general integral formula to compute volumes.

**Common errors to avoid**

- We are dealing with 3D objects, whose visualization may be a problem. Use lots of sketches and have your visual ideas checked by someone else, to avoid misunderstandings.
- Careful in determining the length of the radius in each case: it is the key element to find, so do it well, so as to avoid nightmarish integrals.
Remember that you may be asked to simply set up the integral, since computing it may be very difficult or impossible and may require approximate methods.

**Learning questions for Section I 5-6**

**Review questions:**

1. Explain how the formulae for volumes of revolution by disks and washers are obtained from the method of slicing.

2. Describe how the four steps in the construction of an integral are implemented for the volume of a solid of revolution obtained by disks or washer.

3. Identify the geometric meaning of each portion of a definite integral that represents the volume of a solid of revolution obtained by disks or washer.

**Memory questions:**

1. What is the name of the solids that are obtained by rotating a region around an axis?

2. Which general formula identifies the method of computing volumes by disks?

3. Which general formula identifies the method of computing volumes by washers?

**Computation questions:**

In questions 1-12, set up the integral representing the volume of the solid obtained by rotating the given region around the given axis, then try to compute such integral, if possible.

1. Region bounded by \( y = \frac{5}{x^3 + 1} \) and \( y = 1 \), rotated around the \( x \) axis.

2. Region bounded by \( y = \ln x \), \( y = 0 \), \( x = e \), rotated around the \( x \)-axis.

3. Region in the third quadrant bounded by \( y = x^3 \) and \( y = x \), rotated around \( y = 2 \).

4. Region bounded by \( y = x^3 \), \( y = -x \) and \( y = 1 \), rotated around \( y = -2 \).

5. Region bounded by \( y = \sqrt{x \sin x^2} \), \( y = 0 \), \( \frac{\pi}{6} \leq x \leq \frac{\pi}{2} \), rotated around \( y = -2 \).
6. Region bounded by \( y = \sqrt[4]{x}, \ y = 0, \ x = 0, \ x = 1 \), rotated around:
   a) the \( x \) axis
   b) the line \( y = -1 \)
   c) the line \( y = 1 \)

7. Region bounded by \( y = \sqrt{\tan x}, \ y = 0, \ x = 0, \ x = \frac{\pi}{4} \), rotated around:
   a) the \( x \) axis
   b) the line \( y = -1 \)
   c) the line \( y = 1 \)

8. Region bounded by \( y = 2 - x^2, \ y = 1 - \cos x, \ x = 0, \ x = 1 \), rotated around:
   a) the \( x \) axis
   b) the line \( y = -2 \)
   c) the line \( y = 2 \)

Each of questions 13-16 represents the volume of a solid of revolution. Identify the boundaries of the region whose rotation generates the solid and the axis of rotation. Watch out for the details!

13. \( \int_{-1}^{0} \left[ (3x^2 - x)^2 - (6x - x^2)^2 \right] dx \)

14. \( \pi \int_{a}^{b} (f(x) + c)^2 dx \)

15. \( \int_{2}^{4} x^3 dx \)

Theory questions:

1. In the method of washers, what does the \( dx \) that appears in the integral represent geometrically?

2. In the formula for computing volumes by disks, what does the integrand represent?

3. The method of computing volumes by disks is a special case of the method of slicing. What is the shape of the sections in that case?

4. In the method of finding volumes by washers, which portion of the integral uses the information about any holes in the solid?

5. If a region bounded by the curve \( y = f(x) \) and the \( x \) axis, for \( a \leq x \leq b \), is rotated around the \( x \) axis, which formula describes the volume of the solid so obtained?
6. If a region bounded by the curves \( y = f(x) \) and \( y = g(x) \), \( a \leq x \leq b, \ f \geq g > 0 \), is rotated around the \( x \) axis, which formula describes the volume of the solid so obtained?

7. What is the shape of the solid whose volume can be given by the integral \( 2\pi \int_{0}^{1} x^2 \, dx \)?

**Templated questions:**

1. Use any of the regions described in the *Computation questions* and compute the volume of the solid obtained by rotating it around a different axis in the same direction and on in the perpendicular direction.

2. Choose a finite region and an axis of rotation that does not cross inside the region and set up the integral representing the volume of the solid obtained by rotating the region around the axis. Try to compute such integral.

**What questions do you have for your instructor?**