How to pick the right method to compute volumes

What you need to know already:

- How to compute volumes by slicing, disks, washers and shells.

What you can learn here:

- How to select the most suitable method to compute the volume of a solid.

We now have several methods for computing the volume of a solid, with solids of revolution providing a simple setting. Some of those methods provide alternatives in many situations, so, how can we decide which method to use? Here is a simple, basic strategy.

Strategy for choosing the best method for computing volume

- Work preferably within the $y = f(x)$ setting, and hence switch names of variables if $x = f(y)$ is given or needed.
- Use \textit{disk} method when dealing with a solid of revolution whose axis of rotation is horizontal and part of the boundary
- Use \textit{washer} method when dealing with a solid of revolution whose axis of rotation is horizontal and separate from the region
- Use \textit{shell} method when dealing with a solid of revolution whose axis of rotation is vertical.
- Use the method of \textit{slicing} if the solid is not obtained through a revolution and the cross-sections have a known area.
- If the integral obtained is too complicated, \textit{consider} inverting the function and check if this produces an easier integral.

This little strategy is a suggestion more than anything new, so try to apply it to any volume problems you have available and see how it works.
Example: \( y = \cos \pi x, \quad y = x^2 - \frac{1}{4} \)

Although solving the equation \( \cos \pi x = x^2 - \frac{1}{4} \) is not possible through basic methods, we notice that the parabola opens up and has \( x \) intercepts at \((-0.5, 0)\) and \((0.5, 0)\). The other curve is a stretched cosine, also having \( x \) intercepts at the same two points. A calculator’s suggestion can be used, as long as it is confirmed algebraically. Hence the relevant graphs is:

If we rotate this region around \( x = -2 \), the axis of rotation is vertical, thus suggesting the use of the shells method. The height of each slice and the radius of rotation are, respectively:

\[
h = \cos \pi x - \left( x^2 - \frac{1}{4} \right), \quad r = x - (-2)
\]

Therefore the volume is given by:

\[
V = 2\pi \int_a^b rhdx = \pi \int_{-0.5}^{0.5} \left(2 + x\right)\left(\cos \pi x - x^2 + \frac{1}{4}\right)dx
\]

If we rotate the same region around the line \( y = 2 \), which is a horizontal line outside of the boundary, we need to use the washers method. In this case the outer and inner radii are, respectively:

\[
R = 2 - \left( x^2 - \frac{1}{4} \right), \quad r = 2 - \cos \pi x
\]

Therefore the volume is given by:

\[
V = \pi \int_a^b \left[ R^2 - r^2 \right] dx = \pi \int_{-0.5}^{0.5} \left(2 - \cos \pi x\right)^2 - \left(\frac{9}{4} - x^2\right)^2 dx
\]

Both integrals are computable: can you see quickly which is the easier one?

**Summary**

- There are several methods for computing the volume of a solid: learn to choose the most convenient one in any given case.

**Common errors to avoid**

- Whatever error you will make in the choice will be most likely due to lack of experience, so, practice a lot!
Learning questions for Section I 5-8

Review questions:

1. Explain how to decide whether to use the method of disks, washers or shells to compute the volume of a solid of revolution.

Computation questions:

In questions 1-10, set up the integral representing the volumes of the solids of revolution obtained by rotating the region described there around the given axis. When possible, compute such integral.

1. The region below \( y = \sqrt[3]{x} \), above the \( x \) axis, between \( x=0 \) and \( x=1 \), rotated around:
   a) the \( x \) axis.
   b) \( x = -1 \).
   c) \( y = 1 \).

2. The region below \( y = \sqrt[3]{x} \), above the line \( y=1 \), between \( x=1 \) and \( x=4 \), rotated around:
   a) \( x = 1 \).
   b) the \( x \) axis.
   c) \( y = 2 \).

3. The region below \( y = \sqrt{\tan x} \), above the \( x \) axis, between \( x=0 \) and \( x=\pi/4 \), rotated around:
   a) the \( y \) axis.
   b) \( y = -1 \)
   c) \( y = 1 \)

4. The region below \( y = \tan x \), above the \( x \) axis, between \( x=0 \) and \( x=\pi/4 \), rotated around:
   a) the \( x \) axis.

5. The region bounded by \( f(x) = 10-x^2 + 2x \), \( g(x) = 10-6x+x^2 \), rotated around:
   a) \( y = -1 \)
   b) \( x = 2 \)
   c) \( x = 2 \)

6. The region bounded by \( y = 2-x^2 \) and \( y = 1-\cos x \), between \( x = 0 \) and \( x = 1 \), rotated around:
   a) the \( y \) axis.
   b) \( x = -2 \)
   c) \( y = 2 \)

7. The region in the first quadrant bounded by the curves \( y = x^3 \) and \( y = 6x-x^2 \), rotated around:
   a) \( x = 3 \)
   b) \( y = -1 \)
8. The region bounded by \( y = e^{x/2} \) and \( y = x^2 + \frac{1}{2} \), between \( x = 0 \) and \( x = 1 \), rotated around:
   a) the \( y \)-axis.
   b) \( y = \frac{1}{2} \)
   c) \( y = 2 \)

9. The region bounded by \( y = 2x - x^2 \), \( y = 2x - 1 \), \( x = 1 \), \( x = 2 \), rotated around:
   a) the \( x \)-axis;
   b) the \( y \)-axis;
   c) \( y = 4 \).

10. The region bounded by the curve \( y = e^{3x} \), the \( y \)-axis and the line \( y = e \), rotated around:
    a) the \( x \)-axis
    b) \( x = 1 \)

11. The integral \( \int_{5}^{8} 2\pi (x - 4)^2 e^{\cos x} \, dx \) can be used to represent the volumes of two solids of revolution, one obtained by rotating a region around a horizontal axis, the other obtained by rotating another region around a vertical axis. Describe the region and identify the axis of rotation for both cases. I expect you to justify your choices, but NOT to compute the integral.

**Theory questions:**

1. Which basic geometric solid shape is used in both the washers and shells method to arrive at the general formula?

2. Which method should one use to compute the volume of a doughnut, when served on a usual plate?

**Templated questions:**

1. Choose a finite region and an axis of rotation that does not cross the inside of the region; then set up the integral representing the volume of the solid obtained by rotating the region around the axis. Try to compute such integral.

**What questions do you have for your instructor?**