Chapter 6: Improper and approximate integration *Section 3*

Improper integrals with one discontinuity between limits

What you need to know already:	*What you can learn here:*
➤ How to interpret and compute an improper integral with an issue at one limit.	➤ How to interpret and compute an integral that is improper because of a discontinuity between its limits of integration.

When an integral $\int_a^b f(x)dx$ is improper because $f(x)$ is discontinuous at a single value *c*, where $a < c < b$, we can still use limits, together with the additive property of integrals for which $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$. However, in this case we need to ensure that both new integrals exist, just like in order for a limit to exist, both the left and right limits must exist.

Definition

If $y = f(x)$ is a function that is continuous for $a \le x \le b$, except for a single discontinuity at a value $x=c$ between the limits, the ***improper*** integral $\int_a^b f(x)dx$ can be defined as:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$= \lim_{h \to c^-} \int_a^h f(x)dx + \lim_{k \to c^+} \int_k^b f(x)dx$$

If both limits exist, the integral is ***convergent*** to their sum, otherwise it is ***divergent***.

Notice that the above definition uses two different dummy variables, *h* and *k*, for the two integrals. This is to emphasize that the two integrals must be computed separately. In practice, it is acceptable to use the same letter in both, as long as the two integrals are computed separately and not combined.

Example: $\int_0^4 \frac{1}{(x-2)^{2/3}}dx$

This integral is improper because the integrand is discontinuous at $x=2$, which is between the limits. Therefore we split it into the two pieces:

$$\int_0^4 \frac{1}{(x-2)^{2/3}}dx = \lim_{k\to 2^-}\int_0^k \frac{1}{(x-2)^{2/3}}dx + \lim_{k\to 2^+}\int_k^4 \frac{1}{(x-2)^{2/3}}dx$$

$$= \lim_{k\to 2^-}\left[3(x-2)^{1/3}\right]_0^k + \lim_{k\to 2^+}\left[3(x-2)^{1/3}\right]_k^4 =$$

$$= \lim_{k\to 2^-}\left[3(k-2)^{1/3} + 3\sqrt[3]{2}\right] + \lim_{k\to 2^+}\left[3\sqrt[3]{2} - 3(k-2)^{1/3}\right]$$

$$= 3\sqrt[3]{2} + 3\sqrt[3]{2} = 6\sqrt[3]{2}$$

Since both integrals converge, we conclude that the original integral also converges, to $6\sqrt[3]{2}$.

I will give you just one more example, since you will gain more by trying your own hand with this situation.

Example: $\int_{-2}^2 \frac{e^x}{e^x-2}dx$

This integral is improper because its integrand is discontinuous at $x=\ln 2$. Therefore we need to compute:

$$\lim_{k\to \ln 2^-}\int_{-2}^k \frac{e^x}{e^x-2}dx + \lim_{k\to \ln 2^+}\int_k^2 \frac{e^x}{e^x-2}dx$$

To compute the antiderivative, we use the substitution $u=e^x-2,\ du=e^x dx$ to obtain:

$$\int \frac{e^x}{e^x-2}dx = \int \frac{du}{u} = \ln u + c = \ln\left|e^x-2\right| + c$$

This antiderivative can be used for both integrals, but the limits must still be computed separately. Therefore:

$$\lim_{k\to \ln 2^-}\int_{-2}^k \frac{e^x}{e^x-2}dx + \lim_{k\to \ln 2^+}\int_k^2 \frac{e^x}{e^x-2}dx =$$

$$= \lim_{k\to \ln 2^-}\left[\ln\left|e^x-2\right|\right]_{-2}^k + \lim_{k\to \ln 2^+}\left[\ln\left|e^x-2\right|\right]_k^2$$

Since both limits involve a $\ln 0$, neither converges, hence the whole integral is divergent.

Summary

- When an integral is improper because of a single discontinuity between the limits of integration, we split it at that value and check if both pieces are convergent.

Common errors to avoid

- Do not combine the two integrals when checking for convergence, as that defeats the purpose!
- Remember that if one of the two smaller integrals diverges, the whole integral diverges.

Learning questions for Section I 6-3

Review questions:

1. Explain how to give meaning to an improper integral whose difficulty occurs between the limits of integration.

Memory questions:

1. If $f(x)$ is discontinuous only at $x = c, a < c < b$, what does $\int_a^b f(x)dx$ mean?

Computation questions:

For each of the improper integrals presented in questions 1-10:

- identify why it is improper,
- determine if it is convergent or divergent,
- in case of convergence, compute its value,
- in case of divergence determine the cause.

1. $\int_1^{36} \frac{1}{(x-9)^{2/3}} dx$

2. $\int_0^8 \frac{x}{x^2-4} dx$

3. $\int_{-1}^{1} \frac{e^{3/x}}{x^2} dx$

4. $\int_{-1}^{1} \frac{2}{e^x - 1} dx$

5. $\int_{4}^{13} \frac{dx}{(x-5)^3}$

6. $\int_{0}^{2} \frac{xdx}{\sqrt[3]{1-x^2}}$

7. $\int_{-1}^{1} x^{-\pi} dx$

8. $\int_{-1}^{1} \frac{\pi^{1/x}}{x^2} dx$

9. $\int_{1/e}^{e} \frac{dx}{x \ln x^2}$

10. $\int_{1/e}^{e} \frac{dx}{x(\ln x)^2}$

Proof questions:

1. Given a to be a positive constant, determine whether the integral $\int_{-a}^{a} \frac{x+a}{2ax + x^2} dx$ is convergent.

Templated questions:

1. Construct an improper integral with a single discontinuity between the limits of integration and determine whether it is convergent or divergent.

What questions do you have for your instructor?