Chapter 6: Improper and approximate integration — *Section 4*

Improper integrals with several improper features

What you need to know already:	*What you can learn here:*
➤ How to determine the convergence of an improper integral with a single cause for its being improper.	➤ How to determine the convergence of an improper integral that has finitely many discontinuities and/or infinite limits.

We have seen how to handle an integral that is improper because of a single issue by using limits. We have also seen that when an integral is improper because of a single discontinuity that exists between the limits, we split it up into two pieces and check if BOTH pieces, which now have a single issue each at a limit, are convergent.

Well, we can use the same approach whenever we have more issues that make the integral improper, as long as we only have finitely many issues. More specifically:

Strategy for determining the convergence of a generic improper integral

Assume that $y = f\left(x\right)$ is a function that is continuous for $a \le x \le b$, except for finitely many discontinuities on $\left[a, b\right]$ and/or because one or both

In that case, the improper integral $\int_a^b f(x)dx$ can be given meaning by splitting it into several integrals:

$$\int_a^b f(x)dx =$$

$$= \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \cdots + \int_{c_k}^{b} f(x)dx$$

in such a way that each of the smaller integrals only has either a discontinuity or an infinity at one of the two limits.

If ***all*** the smaller integrals converge, the integral is ***convergent*** to their sum, otherwise it is ***divergent***.

Example: $\displaystyle\int_1^{\infty}\frac{dx}{x^2-9}$

This integral is improper for two reasons:

- It is undefined at $x=3$.
- It has an infinite upper limit

Therefore we split it into three parts:

$$\int_1^{\infty}\frac{dx}{x^2-9}=\int_1^{3}\frac{dx}{x^2-9}+\int_3^{4}\frac{dx}{x^2-9}+\int_4^{\infty}\frac{dx}{x^2-9}$$

Now each of the integrals is improper because of a single issue occurring at one of the limits, so we can use the definitions we have used so far. Therefore our integral can be written as:

$$=\lim_{k\to 3^-}\int_1^{k}\frac{dx}{x^2-9}+\lim_{k\to 3^+}\int_k^{4}\frac{dx}{x^2-9}+\lim_{k\to\infty}\int_4^{k}\frac{dx}{x^2-9}$$

Remember that even though we are being lazy and using the same letter in all three limits, they cannot be combined, but must be assessed separately.

An antiderivative of the integrand is $y=\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|$ (check it!) so that the first integral is:

$$=\lim_{k\to 3^-}\int_1^{k}\frac{dx}{x^2-9}=\frac{1}{6}\lim_{k\to 3^-}\left[\ln\left|\frac{x-3}{x+3}\right|\right]_1^k=$$

$$=\frac{1}{6}\lim_{k\to 3^-}\left[\ln\left|\frac{k-3}{k+3}\right|-\ln\frac{1}{2}\right]=-\infty$$

Since this integral is divergent, the whole integral is divergent.

Notice that the last part of the integral is in fact convergent:

$$\lim_{k\to\infty}\int_4^{k}\frac{dx}{x^2-9}=\frac{1}{6}\lim_{k\to\infty}\left[\ln\left|\frac{x-3}{x+3}\right|\right]_4^k=$$

$$=\frac{1}{6}\lim_{k\to\infty}\left[\ln\left|\frac{k-3}{k+3}\right|-\ln\frac{1}{7}\right]=\frac{1}{6}\lim_{k\to\infty}\left[\ln 1-\ln\frac{1}{7}\right]=\frac{\ln 7}{6}$$

But that is too little: ALL parts must be convergent, and one is not enough

Summary

- To assess the convergence of a generic improper integral, we must split it into several pieces along the limits, so that each piece only has one issue at one of the limits of integration.
- The integral is convergent if and only if each and every piece is convergent.

Common errors to avoid

- Don't ignore any causes of discontinuity, since that may lead to incorrect conclusions.

Learning questions for Section I 6-4

Review questions:

1. Explain how to give meaning to an improper integral with finitely many values that make it such.

Memory questions:

1. What must be done to evaluate an integral that has more than one reason for being improper?

Computation questions:

For each of the improper integrals presented in questions 1-4:

- identify why it is improper,
- determine if it is convergent or divergent
- in case of convergence, compute its value,
- in case of divergence determine the cause.

1. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2-4}\,dx$

2. $\displaystyle\int_{-\infty}^{0} \frac{e^{\frac{1}{x}}}{x^2}\,dx$

3. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+4}\,dx$

4. $\displaystyle\int_{1}^{\infty} \frac{dx}{x\ln^2 x}$

5. For which values of k, if any, does the integral $\int_0^1 \frac{dx}{x(\ln x)^k}$ converge?

Theory questions:

1. Can a bounded function give rise to a divergent improper integral?
2. Why is L'Hospital's rule relevant when computing improper integrals?
3. When is it necessary to split an improper integral to check if it is convergent?
4. How many integrals are needed to assess the convergence of $\int_0^\infty x^{-\pi}\,dx$?
5. What are the two main features that make an integral improper?

Proof questions:

1. Use the substitution $u=\sqrt{\frac{x}{1-x}}$ to prove that $\int_0^1 \sqrt{\frac{x}{1-x}}\ln\left(\frac{x}{1-x}\right)dx=\pi$.

   This statement may puzzle you, since so far we have seen logarithmic functions linked mainly to the number e, while here it is linked to the other, more famous irrational number, that is, π. You will soon see this connection become clearer and stronger.
   Also, proving this fact involves several steps and other integration methods, besides, of course, improper integral. But it only requires methods you have seen and are in your arsenal by now. So, don't give up too soon and think of it as rather challenging proficiency question!

Templated questions:

1. For any improper integral you have to compute, determine each of the reasons why it is improper, describe why it is improper at each of them and how the integral needs to be split in order to attempt its evaluation.

What questions do you have for your instructor?