In the midpoint rule we approximate each slice of the region whose area we are computing by using a straight horizontal line, that is, a constant function. Why not use a different type of functions that may better approximate the original curve?

This looks like a good idea, and we start implementing it by using, for each slice, a linear function that joins left- and right-most top points on that interval. This produces a set of trapezoids, instead of rectangles, as this picture shows.

Since we know the formula that provides the area of a trapezoid, we can apply it, together with some simple algebra that I urge you to try at least once, to arrive at the following method.

**Strategy for approximating definite integrals:**

**The trapezoid method**

If \( f(x) \) is a continuous function on \([a, b]\), one way to approximate the definite integral \( \int_a^b f(x)dx \) is to use intervals of equal length and, for each slice, a slant linear functions joining the beginning and end points of the curve in that slice.

In this way we use:

\[
\Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x, \quad i = 0, 1, 2, \ldots, n
\]

and obtain the trapezoidal formula:
\[
\int_{a}^{b} f(x)\,dx \approx \frac{b-a}{2n} \left[ f(x_0) + f(x_n) + 2\left( f(x_1) + f(x_2) + \cdots + f(x_{n-1}) \right) \right]
\]

One way to remember the formula is to notice that the first and last values of the function are used once, since each of them uses only one trapezoid, while all the middle ones are used twice, since they belong to two trapezoids each.

**Example:** \( \int_{1}^{2} \frac{e^{-x}}{x} \,dx \)

This is another integral that cannot be computed through the FTC, since there is no easy way to get an antiderivative. If we use the trapezoid method with \( n=4 \), we use:

\[ x_0 = 1, x_1 = 1.25, x_2 = 1.5, x_3 = 1.75, x_4 = 2 \]

thus getting:

\[
\int_{1}^{2} \frac{e^{-x}}{x} \,dx \approx \frac{2-1}{2 \times 4} \left[ \frac{e^{-1}}{1} + \frac{e^{-2}}{2} + 2 \left( \frac{e^{-1.25}}{1.25} + \frac{e^{-1.5}}{1.5} + \frac{e^{-1.75}}{1.75} \right) \right] \approx 0.1738
\]

By using a calculator’s estimation function, I got the value of 0.1705 for this integral. Thus the trapezoid estimate is an overestimate. This is explained by the fact that the integrand is concave up between the limits of integration and hence all slant sides of the trapezoids are above the curve.

Notice, however, that we still get a decent approximation, correct to two significant digits.

At first sight this method may seem better, since the approximating top boundary follows the curve more closely. However, by thinking more about it we discover that this method lacks the balancing feature of the midpoint rule.

**Example:** \( \int_{1}^{2} \frac{e^{-x}}{x} \,dx \)

If we use the midpoint method with \( n=4 \), we select:

\[ x_1 = 1.125, x_2 = 1.375, x_3 = 1.625, x_4 = 1.875 \]

thus getting:

\[
\int_{1}^{2} \frac{e^{-x}}{x} \,dx \approx \frac{2-1}{4} \left[ \frac{e^{-1.125}}{1.125} + \frac{e^{-1.375}}{1.375} + \frac{e^{-1.625}}{1.625} + \frac{e^{-1.875}}{1.875} \right] \approx 0.1689
\]

This is closer to the calculator’s estimate of 0.1705, since the midpoints do a better job, in this case, of balancing the pieces above the curve with those below the curve, unlike the trapezoid approximation, which is always above the curve.
The balancing done by the midpoint rule is actually a common and convenient feature of that method, while in the trapezoidal rule, most slices tend to be either entirely under the curve (when the curve is concave down) or above it (when the curve is concave up).

Further analysis, to be seen in a later section, reveals that, in general, this method is worse than the midpoint rule. Still, it is an alternative to be kept in mind, one that can still provide good results. In fact, it can still provide as good an approximation as we want if we take \( n \) to be large enough. Moreover, it has the advantage of not requiring the computation of the midpoint values.

**Summary**

- The trapezoid method is a reasonable alternative to the midpoint method, but it is not as efficient, since it lacks the balancing features of the latter.

**Common errors to avoid**

- Use the formula properly and do not confuse the values of \( x \) needed by the midpoint method with those needed by the trapezoid method.

**Learning questions for Section I 6-6**

**Review questions:**

1. Explain why the trapezoid method may be seen as an improvement on the midpoint method.

2. Describe how the formula for the trapezoid method is constructed.

**Memory questions:**

1. Which formula represents the trapezoidal approximation to an integral?

2. What type of curves is used to approximate the top boundary of each strip in the trapezoidal rule?
**Computation questions:**

For each of the integrals provided in questions 1-6, evaluate the integral exactly and then estimate it by using the trapezoidal method with $n=6$.

1. $\int_1^4 \ln \sqrt{x} \, dx$

2. $\int_0^2 e^{-x^2} \, dx$

3. $\int_0^9 \sqrt{5 - \sqrt{x}} \, dx$

4. $\int_0^3 xe^{x^2} \, dx$

5. $\int_1^4 \frac{\pi^{1/x}}{x^2} \, dx$

6. $\int_1^4 x^{-\pi} \, dx$

7. Use the trapezoidal rule to approximate the area of a quarter circle of radius 2 with $n=4$ and compare it to the exact value.

8. Use the trapezoidal rule to estimate the value of $\int_0^a e^{-x^2} \, dx$. Notice that this integral cannot be computed exactly, but notice also that this integral is related to the probability values of a normal (Gaussian, bell-shaped) distribution.

9. Set up, but do not compute, the trapezoidal formula for the integral $\int_0^\pi \sin(\sin(x)) \, dx$ with $n=6$.

10. Compute the trapezoidal approximation to $\int_0^6 \sin \sqrt{x} \, dx$ with $n=6$ and compare it to the value of 5.048... provided by my TI83 and to the value of 5.113 provided by the midpoint formula.

**Theory questions:**

1. If the second derivative of the integrand is positive, what is guaranteed about the trapezoid estimate?

2. What limitation on the integrand is required by all methods of approximate integration?

**Templated questions:**

1. Construct a simple definite integral and use the midpoint rule to estimate its value.

2. For any given definite integral, use the graph of the integrand to determine whether the trapezoidal approximation provides an overestimate, or an underestimate, or whether this is unclear.