**Chapter 6: Improper and approximate integration** | **Section 7**

# Approximate integration:

# Simpson's method

| *What you need to know already:* | *What you can learn here:* |
| --- | --- |
| ➤ What the midpoint and trapezoidal rules are and how to use them. | ➤ Another method for approximating the value of a definite integral. |

The midpoint rule provides a simple and effective way to estimate a definite integral and can be made as accurate as we want. So does the trapezoid rule, although in a way that turns out to be less efficient.

But the idea of using curved top boundaries to better approximate the needed area is still appealing and worth exploring a little more.

Therefore, our next attempt will be to try using quadratic functions, that is, parabolas, as top boundaries. Since in order to identify a parabola we need three points, it follows that in order to carry out this approximation we need to divide the region in ***pairs*** of slices and hence that we need $n$ to be even.

Since we use curved tops for our slices, this approximation should work well and avoid the one-sidedness of the trapezoid method (its being over or under in a consistent manner), but is the corresponding formula too complicated? And will it be as much of a disappointment as the trapezoid method, maybe for other reasons that we do not see yet?

It turns out that quadratic functions fit well with integration from the computational point of view, so that, once all simplifications are done, the final formula is fairly reasonable. Such formula is credited to a British mathematician who popularized it, not after the cartoon series of the same name! ☺ However, it had been used by earlier scientists and is also known in some places as *Kepler's formula*.

## Strategy for approximating definite integrals:

## Simpson's rule

The definite integral of a continuous function is well approximated by the formula:

$$\int_a^b f(x)dx \approx \frac{b-a}{3n}\Big[f(x_0)+$$
$$+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots$$
$$\cdots+4f(x_{n-1})+f(x_n)\Big]$$

where $n$ is an even integer and, as for the trapezoid method:

$$\Delta x=\frac{b-a}{n},\ x_i=a+i\Delta x,\ i=0,1,2,\ldots,n.$$

**Example:** $\int_1^2 \frac{e^{-x}}{x}dx$

In the section on the trapezoid method, we approximated this integral, with $n=4$ and obtained a value of 0.1738, which was an overestimate when compared with a more accurate value of 0.1705 provided by the better estimate of a calculator's function.

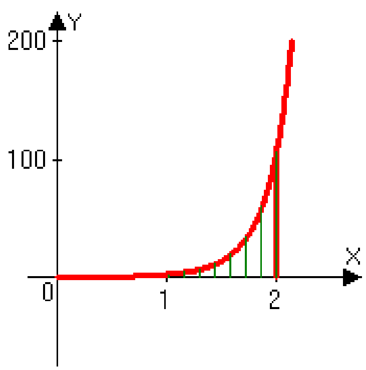
If we use Simpson's formula with *n*=4, we get:

$$\int_1^2 \frac{e^{-x}}{x}dx \approx$$

$$\approx \frac{2-1}{12}\left[\frac{e^{-1}}{1}+4\frac{e^{-1.25}}{1.25}+2\frac{e^{-1.5}}{1.5}+4\frac{e^{-1.75}}{1.75}+\frac{e^{-2}}{2}\right] \approx 0.1706$$

This is a pretty good estimate, being off by 0.0032 or, in percentage terms, by $\frac{0.0032}{0.1705}100 \approx 1.9\%$ . But do other examples fare as well?

**Example:** $\int_0^2 xe^{x^2}dx$




It would seem that this curve resembles a parabola, so Simpson;s rule should work well, right?

Well, we can compute this integral exactly by using substitution and the FTC:

$$\int_0^2 xe^{x^2}dx = \left[\frac{1}{2}e^{x^2}\right]_0^2 = \frac{1}{2}e^4 - \frac{1}{2} \approx 26.799$$

Let us now use Simpson's rule with *n*=6 and assess the corresponding error.

In this case, $\Delta x = \frac{1}{3},\ \ x_i = 0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2$ and we have:

$$\int_0^2 xe^{x^2}dx \approx$$

$$\frac{1}{9}\left[0e^0 + 4\frac{1}{3}e^{1/9} + 2\frac{2}{3}e^{4/9} + 4e^1 + 2\frac{4}{3}e^{16/9} + 4\frac{5}{3}e^{25/9} + 2e^4\right]$$

$$\approx 27.404$$

Therefore, we are off by approximately $27.404 - 26.799 = 0.605$, with a percentage error of:

$$p = \frac{0.605}{26.799}100 = 2.26\%$$

Still quite good.

Are these estimates better because the method is better or simply because the functions do not curve much in the given interval? You may want to try a few more exercises on your own before looking at the next section, where I will present a formal answer to the question.

*But where does the formula come from?*

An Easter egg! Seriously, this is a very good question and the proof is not difficult and you can try it yourself by following the directions in one of the *Learning questions*.

## Summary

- Simpson's method uses parabolas (quadratic functions) to approximate a given integral.
- It seems to perform well in general, but a more objective and formal assessment is needed.

## Common errors to avoid

- Don't get frustrated by the calculations: remember that this is what computers are for!

## Learning questions for Section I 6-7

### Review questions:

1. Explain why Simpson's method may be seen as an improvement on the midpoint method.
2. Describe how the formula for Simpson's rule is structured.

### Memory questions:

1. Which formula represents the Simpson approximation to an integral?
2. What type of curves is used to approximate the top boundary of each strip in the Simpson rule?
3. What limitation on the number of intervals is required by Simpson's rule?

## ***Computation questions:***

Use Simpson's rule with $n$=6 to estimate the value of each of the definite integrals presented in questions 1-10. If feasible, compute also its values obtained by using the FTC, the midpoint and trapezoidal rules and a calculator or computer program and compare them all.

1. $\int_3^9 \sqrt{5-\sqrt{x}}\,dx$

2. $\int_1^4 \frac{\ln x}{x}\,dx$

3. $\int_1^4 \ln\sqrt{x}\,dx$

4. $\int_1^4 \frac{\pi^{1/x}}{x^2}\,dx$

5. $\int_0^{\pi} \sin(\sin(x))\,dx$

6. $\int_1^4 x^{-\pi}\,dx$

7. $\int_0^6 \sin\sqrt{x}\,dx$

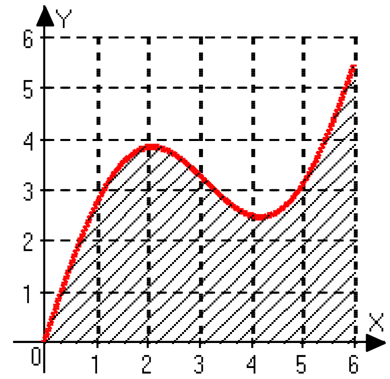
8. $\int_2^4 \frac{x}{\ln^2 x}\,dx$

9. $\int_1^4 \ln\left(x+x^2\right)dx$

10. $\int_1^4 \sqrt{x+x^3}\,dx$

11. Set up the integral that represents the length of the curve $y = x + \ln x, 1 \le x \le 4$ and then the formula that represents its Simpson approximation for $n$=6. No need to actually compute either formula.

12. Use Simpson's rule to estimate the area of the region shaded in the graph shown here.




13. Use Simpson's rule to approximate the area of a quarter circle of radius 2 with $n$=4 and compare it to the exact value.

## ***Theory questions:***

1. What limitation on the integrand is required by all methods of approximate integration?

2. What feature of an integrand may lead to Simpson's estimate being close to the trapezoid estimate?

3. Why is it necessary to choose *n* even when using Simpson's rule?

4. Generally, for what kind of curves can we expect Simpson's to provide a good estimate?

5. How can we use Simpson's method so as to achieve a required precision in the estimate?

6. If an integral represents an arc length, but cannot be computed by using the Fundamental Theorem of Calculus, can one still approximate its value by using Simpson's rule?

## ***Proof questions:***

1. Prove that Simpson's rule produces not just an approximation, but the actual exact value when $y = f(x)$ is a polynomial of degree 2 or 3.

2. Prove that Simpson's rule is correct by doing the following:
   a) Construct the formula for the quadratic function $y = ax^2 + bx + c$ that contains three given points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ with $x_3 = x_2 + \Delta x = x_1 + 2\Delta x$. That is, find the coefficients of the function in terms of the coordinates.
   b) Use the above version of the function to compute $\int_{x_1}^{x_3} (ax^2 + bx + c)\,dx$, again in terms of the coordinates of the points.
   c) Approximate a general definite integral by dividing its interval of integration into pairs of slices of equal length and applying to the three points of the curve that identify each pair the formula developed in part b).
   d) Make all needed algebraic simplifications to arrive at Simpson's formula.

## ***Application questions:***

1. Construct Simpson's formula with *n*=6 to approximate the area of the surface obtained by rotating the arc $y = x\ln x, 1 \le x \le 4$ around the *x* axis.

## ***Templated questions:***

1. Construct a simple definite integral and use Simpson's rule to estimate its value.

## ***What questions do you have for your instructor?***