Chapter 6: Improper and approximate integration | Section 8

Approximate integration:

Comparing approximate integration methods

What you need to know already:	What you can learn here:
➤ The midpoint, trapezoidal and Simpson's rules.	➤ How these methods perform and compare to one another.

This is a mostly theoretical section and even the practice questions will only be of the theoretical persuasion. I only want to give a quick answer to the question of whether the different performances we have observed from the three methods of approximation can be justified more formally.

Is the trapezoid method really better than the midpoint, in some sense? Is Simpson's rule to be preferred to both?

There is no definitive answer that can be applied to all integrals, but there is a qualitative answer that may shed some light.

Technical fact

Assume that $f(x)$ is a ***twice differentiable*** function for which we want to approximate the value of $\int_a^b f(x)dx$. If we denote by:

- ***I*** the actual value of the integral
- ***M*** the midpoint approximation
- ***T*** the trapezoid approximation
- ***S*** the Simpson's approximation
- ***k*** the maximum of $|f''(x)|$ between a and b

then the size of the error of each approximation is given by, respectively:

$$|M-I| \le \frac{k(b-a)^3}{24n^2}$$

$$|T-I| \le \frac{k(b-a)^3}{12n^2}$$

$$|S-I| \le \frac{k(b-a)^5}{180n^4}$$

Notice the following qualitative aspects of these formulae.

1) They provide upper bounds for the possible errors, that is, they tell us how big the errors can be in general, but do not tell us how big they actually are in each case.
2) All formulae reveal that the approximations are worse when the second derivative is large, that is, when the original function curves a lot.
3) The upper bounds for the midpoint and trapezoidal rules look alike, except for the fact that the denominator of the midpoint formula is twice as big, resulting in an error bound twice as small. This confirms that the trapezoidal rule does tend to perform worse than the midpoint rule.
4) The upper bound formula for Simpson's rule has a bigger power for n in the denominator and also a larger coefficient there. This confirms that this method tends to perform better (larger denominator implies smaller error), especially when n is large. So, this method gets us closer to the true value with smaller values of n.

Can we construct an even better procedure by using higher degree polynomials as top boundaries, such as cubic or higher?

It turns out that we can, but the resulting formulae become too unwieldy to be used effectively. However mathematicians have developed other methods based on different concepts; these are the ones used in your calculators to evaluate, for instance, transcendental functions and their integrals.

Summary

- The size of the estimate of each of the three methods we have seen has an upper bound that depends on the number of intervals chosen and the size of the second derivative, that is, how much the curve curves.
- According to these formulae, Simpson's rule does tend to do best of the three and the midpoint tend to do better than trapezoid. However, in individual cases this may be different.

Common errors to avoid

- Remember that the proofs of these formulae are rather complicated and we use the formulae only to have a qualitative assessment of the performance of the methods. Don't worry about understanding where the formulae come from or how to use them.

Learning questions for Section I 6-8

Review questions:

1. Explain how the three man methods of approximate integration are compared in terms of their general performance.

Memory questions:

1. Generally, which of the three main approximation methods provides the best estimates?

Computation questions:

1. Use all three methods to estimate $\int_0^1 \left(x - x^3\right) dx$ and compare the values obtained to the exact value: what do you observe?
2. Use all three methods applied to the integral $\int_1^7 \frac{dx}{x}$ to estimate the value of $\ln 7$ and compare the values obtained to the value provided by your calculator: what do you observe?
3. Use all three methods applied to the integral $\frac{1}{\sqrt{2\pi}} \int_0^1 e^{-x^2} dx$ to estimate the value of the probability for a standard normal random variable to be between 0 and 1.

Theory questions:

1. Why do we need to consider methods such as Simpson's?
2. Why don't we use a formula similar to Simpson's but based on cubic polynomials?
3. Do the approximation formulae we have studied err more if the graph of the integrand curves a lot or if it curves little?
4. What limitation on the integrand is required by all methods of approximate integration?
5. Which approximation method can be used to estimate the value of a convergent integral that is improper because the integrand is undefined at one of its limits?

What questions do you have for your instructor?