Chapter 1: Geometric vectors **Section 2**

Definition of geometric vectors

What you need to know already:	What you can learn here:
➤ The general aims behind the concept of a vector.	➤ The formal definition of the traditional two- and three-dimensional vectors used in physics and known as *geometric vectors*. ➤ The basic notation and terminology used for geometric vectors.

We start simple, by defining 2-dimensional vectors, the simplest vectors there are.

Are these the familiar arrows in a Cartesian plane?

Yes and no! They are commonly represented as arrows in a 2-dimensional Cartesian plane, but we shall look at them from a more formal definition that will allow us to generalize the notion later.

Definition

A ***2-dimensional vector*** (or simply a ***2D vector***) consists of an ordered ***pair*** of scalars a and b called ***components***.

Like speed and direction?

It can be done that way, but we shall use a different approach, one that makes use of something that you have used umpteen times before, namely the coordinates of a point in the Cartesian *x*-*y* plane.

Are you saying that a vector with components a *and* b *is the same as a point* (a, b)*?*

I said that we shall *use* coordinates to give meaning to components, ***not*** that they are the same thing. To emphasize the connection, but also to see the differences, we shall follow this convention in the notation.

Definition

A 2D vector with components a and b will be ***denoted*** by a lower case bold letter and its components will be listed in square brackets, as in:

$$\mathbf{v} = [a \;\; b]$$

Knot on your finger

To denote the components of a generic vector **v** we may also use subscripts, as in:

$$\mathbf{v} = [v_1 \;\; v_2]$$

We shall use a boldface, lower case letter (**v**) to represent a vector, as opposed to the italic capital letters (P) commonly used to represent a point. Also, we report the two components in square brackets, as $[a \;\; b]$ instead of the round brackets notation (a, b) used for coordinates. Finally, notice that whenever no confusion is created we shall separate the components by a space rather than the comma that is traditional with coordinates.

Example:

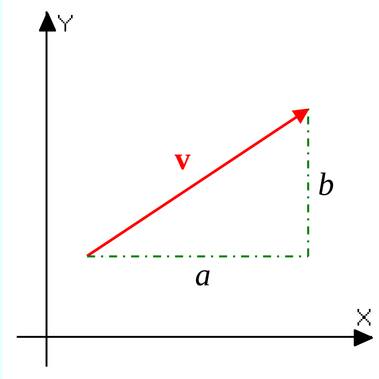
The ordered pairs $[1 \;\; 2]$, $\begin{bmatrix} \frac{2}{5} & -4 \end{bmatrix}$, $[-8 \;\; \pi]$ are all 2D vectors, as well as any other pair of numbers you can imagine. But remember that the vector $[1 \;\; 2]$ is not the same as the vector $[2 \;\; 1]$, since the order in which the numbers are listed is important in a vector, just as it is for the coordinates of a point. And keep in mind that the vector $[1 \;\; 2]$ is not the same as the point $P(1, 2)$.

I am confused: if vectors are not points, what is the connection between the two and where is the geometry?

One clarification coming up:

Definition

A two dimensional vector $\mathbf{v} = [a \;\; b]$ can be represented geometrically as a ***directed segment***, that is, an ***arrow*** whose run is a and whose rise is b, as shown in the picture.

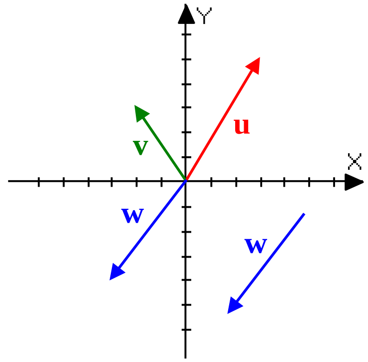


We shall refer to the starting point of the arrow representing a 2D vector as its ***tail*** and its end point as its ***tip***.

A 2D vector **v** can be represented by an arrow with tail at any point, depending on the ***application***, and sometimes there is no need to specify the tail.

Example:

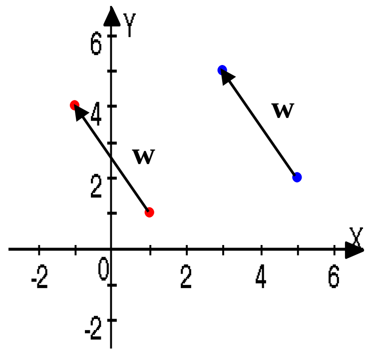
Here you can see a geometric representation of the 2D vectors $\mathbf{u} = [3 \;\; 5]$, $\mathbf{v} = [-2 \;\; 3]$ and $\mathbf{w} = [-3 \;\; -4]$, all with tail at the origin. You can also see another representation of the vector **w** whose tail is at some other point.



Again remember that while you may be used to thinking of these arrows as *being* the vectors, you need to start thinking of them simply as geometrical or graphical *representations* of the corresponding vectors.

Example:

The vector $\mathbf{w} = [-2 \;\; 3]$ can be represented by the arrow with tail at $(1, 1)$ and tip at $(-1, 4)$, or by the arrow with tail at $(5, 2)$ and tip at $(3, 5)$, as shown in the picture. Once again, each of these arrows is a *representation* of the vector, not the vector itself.



The following technical fact, which you have probably seen before, is an immediate consequence of the definition of a 2D vector and you can see an illustration of it in the last example.

Technical fact

If the points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are the tail and tip respectively of an arrow representing the vector $\mathbf{v} = [v_1 \;\; v_2]$, then:

$$v_1 = x_2 - x_1, \qquad v_2 = y_2 - y_1$$

In the same way we have that:

$$x_2 = x_1 + v_1, \qquad y_2 = y_1 + v_2$$

Definition

Given two points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$, the vector $\mathbf{PQ}$ is the one represented by the arrow with P as tail and Q as tip:

$$\mathbf{PQ} = [q_1 - p_1 \;\; q_2 - p_2]$$

Given a point $P = (p_1, p_2)$, the vector with components $[p_1 \;\; p_2]$ can be represented by an arrow with tail at the origin O and tip at P and is usually denoted by $\mathbf{P}$.

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This ***overlap*** of notations and ***variety of fonts*** can generate significant simplifications in most cases, but also some confusion, so that it should be clarified whenever needed.

I can see that! It will take some time to get used to such variety and potential confusion.

It will, but nothing that a suitable amount of practice and focus cannot fix.

Speaking of variety of notation, I remember seeing vectors denoted with an upper arrow, like $\vec{\mathbf{v}}$. *Is that wrong?*

Certainly not, but that notation is strictly related to the arrow representation of a vector, which is just one of the possible uses of vectors and will be ignored in many other uses.

In fact, there are other notations that are used in different books and by different people to represent vectors. Here is a sample of such alternatives, in case you find them somewhere else.

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A 3D geometric vector $\mathbf{v} = [v_1\ v_2\ v_3]$ may also be indicated through one of the following notations:

$$\overline{v},\ \underline{v}, \vec{v},\ \underline{\mathbf{v}},\ \vec{\mathbf{v}}$$

$$[v_1, v_2, v_3], (v_1\ v_2\ v_3), (v_1, v_2, v_3)$$

$$\langle v_1\ v_2\ v_3 \rangle, \langle v_1, v_2, v_3 \rangle$$

Other special notations are also possible and used. The same applies to 2D vectors.

So, will you use these alternatives?

I will not use the upper and lower bars or the upper arrow, both to save ink (☺) and because I find them redundant when it is clear that we are dealing with vectors.

Also, I will not use the other component notations because they are easily confused with other symbols, such as coordinates. Finally, the square brackets notation will prove later to be consistent with the notation used for matrices, a fact that will turn out to be very useful.

However, feel free to use these alternative notations and be prepared to identify and understand them when they are used by other people.

Before we move on, here is a definition that may seem pathetically simple now, but will be used extensively later, both to simplify our discussions and to eliminate pedantic little problems.

Definition

🗣 The vector $\mathbf{0} = [0\ \ 0]$ is called the ***zero vector*** (duh!) and any other vector is said to be a ***non-zero vector***.

Now that we have looked at the most basic vectors, let us step up a bit and make the acquaintance of 3D vectors.

I suspect that we just need to extend the concepts we have seen so far to 3-dimensional space, right?

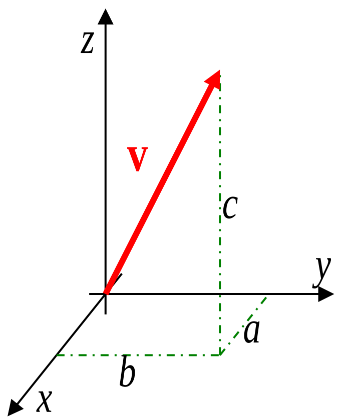
Of course! The only problem is that the pictures become more difficult to draw on a 2-dimensional piece of paper. But the concepts are the same:

Definition

A ***3-dimensional (3D) vector*** is an ordered triple of scalars, called components.

A 3D vector is ***denoted*** by $\mathbf{v} = [a\ \ b\ \ c]$ or $\mathbf{v} = [v_1\ \ v_2\ \ v_3]$ and can be represented by a directed arrow in 3-dimensional space having a, b and c as the corresponding lengths in the x, y and z directions respectively.

All terminology related to the arrow representation of 2D vectors ***extends*** to 3D vectors, including the concepts of ***zero*** and ***non-zero*** vectors.



Here is a representation of a 3D vector $\mathbf{v} = \begin{bmatrix} a & b & c \end{bmatrix}$ with tail at the origin. Keep in mind that because of problems with perspective, it is difficult to draw and to see a 3D vector whose tail is not at the origin, but fortunately, we shall not have to do this often.

Hmmm, I can see why the pictures are more difficult to visualize…

…and to draw! So, as much as possible we'll stick to 2D vectors for illustration and leave the rest to your visual imagination.

The following technical fact is the obvious extension of the same fact I stated for 2D vectors, so I am repeating it here to emphasize its existence and usefulness.

Technical fact

If the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are the tail and tip respectively of a geometrical representation of the vector $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$, then:

$$x_2 - x_1 = v_1, \quad y_2 - y_1 = v_2, \quad z_2 - z_1 = v_3$$

In the same way:

$$x_2 = x_1 + v_1, \quad y_2 = y_1 + v_2, \quad z_2 = z_1 + v_3$$

Definition

Given two ***points*** $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$, the ***vector*** $\mathbf{PQ}$ is the one represented by the arrow with P as tail and Q as tip:

$$\mathbf{PQ} = \begin{bmatrix} q_1 - p_1 & q_2 - p_2 & q_3 - p_3 \end{bmatrix}$$

Given a ***point*** $P = (p_1, p_2, p_3)$, the ***vector*** $\begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$ represented by the arrow with the origin O as tail and P as tip can also be denoted by $\mathbf{P}$.

Knot on your finger

This ***overlap*** of notations and ***variety of fonts*** can generate significant simplifications in most cases, but also some confusion, so that it should be clarified whenever needed.

By the way, this overlap may create confusion and occasional errors for teachers as well! Point them out when you see them.

Summary

- A geometric vector is simply an ordered set of 2 or 3 numbers.
- A geometric vector can be represented by an arrow in the plane (2D) or in space (3D) but is not the same as such an arrow.
- Several notations exist to represent vectors and their components; we shall use the one that employs square brackets and no commas.

Common errors to avoid

- Do not confuse a vector (general concept) with an arrow (specific use).
- Do not confuse points with the arrow representation of the vector with its coordinates as components. They are related , but not the same.

Learning questions for Section LA 1-2

Review questions:

1. Explain the difference between the informal description of a vector presented in the previous section and the formal definition provided in this section.
2. Explain in what way a point and a vector are different and in what way they are related.
3. Present some alternative notations commonly used to indicate the components of a 3D vector $\mathbf{v}=\begin{bmatrix} a & b & c \end{bmatrix}$.

Memory questions:

1. What is a 2D vector?
2. What is a 3D vector?
3. What is the name of numbers that make up a geometric vector?

Computation questions:

Draw an arrow representing each of the 2D vectors presented in questions 1-4. Then draw another arrow representing the same vectors, but with a different tail.

1. $\mathbf{u} = \begin{bmatrix} -2 & 5 \end{bmatrix}$

2. $\mathbf{v} = \begin{bmatrix} 1 & \pi \end{bmatrix}$

3. $\mathbf{w} = \begin{bmatrix} \frac{7}{4} & -\frac{5}{2} \end{bmatrix}$

4. $\mathbf{y} = \begin{bmatrix} 0.3 & -2.17 \end{bmatrix}$

Draw an arrow representing each of the 3D vectors presented in questions 5-8.

5. $\mathbf{u} = \begin{bmatrix} -2 & 5 & -1 \end{bmatrix}$

6. $\mathbf{v} = \begin{bmatrix} 1 & \pi & -1 \end{bmatrix}$

7. $\mathbf{w} = \begin{bmatrix} \frac{7}{4} & -\frac{5}{2} & \frac{1}{3} \end{bmatrix}$

8. $\mathbf{y} = \begin{bmatrix} 0.3 & -2.17 & 1.2 \end{bmatrix}$

9. Determine the tail of the geometric representation of the vector $\mathbf{v} = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}$ whose tip is at $(-1, 6, 2)$.

10. Determine the tip of the geometric representation of the vector $\mathbf{v} = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}$ whose tail is at $(-1, 6, 2)$.

Theory questions:

1. When we identify a point with a vector in $\mathbf{R}^3$, what do we assume about the vector?

2. What are the components of a 2D vector?

3. Do the pairs [1 2] and [2 1] represent the same vector?

4. Do the pairs [1 2 3] and [2 1 3] represent the same vector?

5. What is the difference between the vector [1 2] and the point (1, 2)?

6. What is the difference between the vector [1 2 3] and the point (1, 2, 3)?

7. Is a 2D vector a directed segment?

8. Is a 3D vector a directed segment?

9. What is the tip of an arrow representation of a vector?

10. What does an arrow representing the zero vector look like?

Proof questions:

1. Explain why the formulae that link the components of a vector and the coordinates of the tip and tail of one of its arrows are correct.

2. Determine the formulae that provide the coordinates of the tail of the arrow whose tip is at $P_2 = (x_2, y_2, z_2)$ and that represents the vector $\mathbf{v} = [a \;\; b \;\; c]$. Such formulae are needed in *Computation question* 9.

Templated questions:

In these questions, make your own choice of the vector **v**, either 2D or 3D, and the points *P* and *Q*.

1. Determine the tip of the arrow representing the vector **v** with the tail at *P*.

2. Determine the tail of the arrow representing the vector **v** with the tip at *P*.

3. Determine the vector **v** corresponding to the arrow tail at *P* and tip *Q*.

What questions do you have for your instructor?