Chapter 1: Geometric vectors — **Section 3**

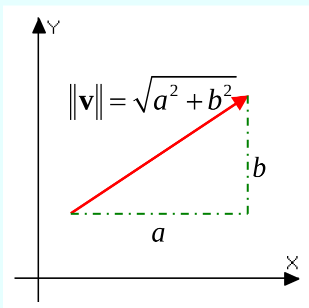
Length of a geometric vector

What you need to know already:	*What you can learn here:*
➤ The formal definition of a geometric vector.	➤ The formal definition of the length of a vector. ➤ The basic notation and terminology associated with the length of a vector.

As I said in the first section, vectors are used whenever we need to identify an entity through more than one quantity, hence the idea of a vector having 2 or 3 components. And as you have seen in earlier courses, we can say that a geometric vector is characterized by having a size, represented by the length of a corresponding arrow, and a direction. Let us start by looking at the length.

Definition

The ***length*** of a 2D vector $\mathbf{v} = [a \;\; b]$ is provided by the Pythagorean theorem and is denoted by enclosing the symbol for the vector between two double bars:



$$\|\mathbf{v}\| = \|[a \;\; b]\| = \sqrt{a^2 + b^2}$$

The ***length*** of a 3D vector $\mathbf{v} = [a \;\; b \;\; c]$ is also obtained through the Pythagorean theorem through the similar formula:

$$\|\mathbf{v}\| = \|[a \;\; b \;\; c]\| = \sqrt{a^2 + b^2 + c^2}$$

Example:

The length of the vector $\mathbf{u} = [3 \;\; 4]$ is given by $\|\mathbf{u}\| = \sqrt{3^2 + 4^2} = 5$.

The length of the vector $\mathbf{a} = [-2 \;\; 1]$ is given by $\|\mathbf{a}\| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$

This double bar notation reminds me of the absolute value…

As it should, since it comes from it, but the bars are doubled to indicate that we are dealing with vectors and not just scalars. Here are two important similarities between the absolute value and the length of a vector:

Technical facts

Since the ***absolute value*** of a real number is given by the formula $|c| = \sqrt{c^2}$, it can be viewed as the length of a "*1-dimensional vector*".

The length of a geometric vector, just like the absolute value, is ***always positive***, since it is obtained as a sum of squares.

There are other similarities between the absolute value of a number and the length of a vector and we shall explore them as we discover more and more about vectors. But now let me introduce to you a special type of vectors that will play a major supporting role in what follows.

Definition

A ***unit vector*** is a vector whose length is 1.

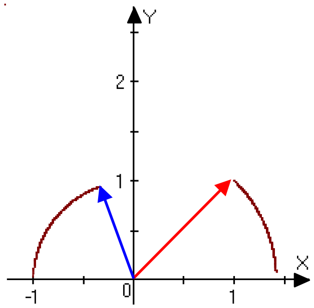
Don't you mean a geometric unit vector?

Good eyes, but I was not being lazy. While it is true that for now we are discussing geometric vectors, this definition of unit vector does not depend on the geometric interpretation and it will apply also to more general vectors. Here are some examples:

Example:

The vector $\begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \end{bmatrix}$ is a unit vector, as $\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\sqrt{\frac{2}{3}}\right)^2} = \sqrt{\frac{1}{3}+\frac{2}{3}} = 1$.

The vector $\begin{bmatrix} 1 & 1 \end{bmatrix}$ shown in the picture may look like a unit vector, since it has lots of 1's, but its length is $\|v\| = \sqrt{1^2+1^2} = \sqrt{2}$ and hence it is not.



On the other hand, the vector $\begin{bmatrix} -\frac{1}{3} & \frac{2\sqrt{2}}{3} \end{bmatrix}$, weird as it may seem, is a unit vector, since $\sqrt{\left(-\frac{1}{3}\right)^2 + \left(\frac{2\sqrt{2}}{3}\right)^2} = \sqrt{\frac{1}{9}+\frac{8}{9}} = 1$.

Definition

The vectors $\begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \end{bmatrix}$ are called the ***standard 2D unit vectors***.

The vectors $\mathbf{i} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $\mathbf{j} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, $\mathbf{k} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ are called the ***standard*** 3D unit vectors.

I have seen these extremely simple unit vectors before, but how important are they?

Important enough that they deserve these special names! We shall indeed see and use them often in later sections.

Summary

- The length of a geometric vector is the length of any one of its arrow representations.
- A unit vector is a vector whose length is 1.

Common errors to avoid

- Watch the arithmetic when applying the formula!

Learning questions for Section LA 1-3

Review questions:

1. Describe what the length of a geometric vector is and how it is computed.
2. Explain what a unit vector is and why such a concept is important.

Memory questions:

1. What is the length of the zero vector?
2. How is the length of a geometric vector **v** denoted?
3. What is a unit vector?
4. What numbers can be components of a unit vector?
5. Which vector does **i** represent?
6. Which vector does **j** represent?
7. Which vector does **k** represent?

Computation questions:

Compute the length of the vectors provided in questions 1-4 and identify those that are unit vectors.

1. $\mathbf{v} = \begin{bmatrix} 3 & -5 \end{bmatrix}$

2. $\mathbf{v} = \begin{bmatrix} \pi & 5.2 & \sqrt{3} \end{bmatrix}$

3. $\mathbf{v} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{2} \end{bmatrix}$

4. $\mathbf{v} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

Theory questions:

1. What is the main purpose of unit vectors?

2. What is the geometric effect of multiplying a vector by a scalar?

Proof questions:

1. Show that the formula for the length of a 3D vector **v** does in fact represent the length of one of its representative arrows. That is, show that the Pythagorean theorem extends to three dimensions.

2. Show that $|c| = \sqrt{c^2}$ for any real number c.

3. Explain why if $[a \quad b \quad c]$ is a unit vector so is $[b \quad c \quad a]$.

4. Explain why if $[a \quad b]$ is a unit vector, so is $[-a \quad -b]$.

5. Determine the value of c so that $\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & c \end{bmatrix}$ is a unit vector.

Templated questions:

1. Compute the length of a 2D vector **v**.

2. Compute the length of a 3D vector **v**.

3. Check if the given 2D or 3D vector **v** is a unit vector.

What questions do you have for your instructor?