Chapter 1: Geometric vectors *Section 4*

Geometric direction of a vector

What you need to know already:	*What you can learn here:*
➤ Definition of a geometric vector. ➤ Length of a geometric vector.	➤ How to identify the direction of a vector in a geometric way. ➤ Why this geometric approach is correct and traditional, but not efficient.

So, the Pythagorean Theorem gives us the length of a vector, but what about its direction?

In calculus you have worked extensively with a quantity that contains information about direction: do you remember what it is?

Doesn't the slope indicate a direction?

Yes, and in fact in 2D we could use this familiar concept to determine the direction of a vector, since the two components do correspond, respectively, to the rise and run of an arrow representing the vector. However, it turns out that the slope has a few peculiarities that make it unsuitable to identify the direction of a vector in general.

For instance, remember that, unlike lines, vectors also have an orientation, meaning that a vector is not just a segment, but an *oriented* segment, an arrow with a tail and a tip. Also, vertical lines do not have a slope, and yet vertical vectors do exist. If we want to take all this into consideration, we can try the following approach.

Definition

Two non-zero 2D vectors $\mathbf{u}=[u_1 \;\; u_2]$ and $\mathbf{v}=[v_1 \; v_2]$ have the same ***direction*** if their slopes are the same or are both undefined:

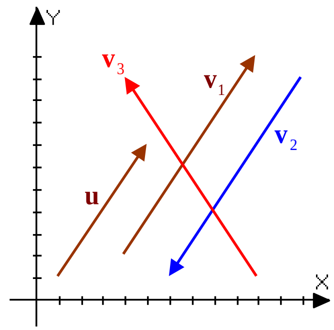
$$m=\frac{u_2}{u_1}=\frac{v_2}{v_1} \text{ or } u_1=v_1=0$$

This is equivalent to the condition $u_2v_1=u_1v_2$.

Two non-zero 2D vectors $\mathbf{u}=[u_1 \;\; u_2]$ and $\mathbf{v}=[v_1 \; v_2]$ that have the same direction are said to also have the same ***orientation*** if $u_1v_1 \geq 0$ and $u_2v_2 \geq 0$.

Example:

The vectors $\mathbf{u}=[4\ \ 6]$ and $\mathbf{v}_1=[6\ \ 9]$ have the same direction, since $\frac{6}{4}=\frac{9}{6}$, and they also have the same orientation, since all components are positive.



The vectors $\mathbf{u}=[4\ \ 6]$ and $\mathbf{v}_2=[-6\ \ -9]$ also have the same direction, since $\frac{6}{4}=\frac{-9}{-6}$, but they have opposite orientations, since the first and second pairs of components have opposite sign and hence their product is negative.

The vectors $\mathbf{u}=[4\ \ 6]$ and $\mathbf{v}_3=[-6\ \ 9]$ do not have the same direction, since $\frac{6}{4}\neq-\frac{9}{6}$.

Notice that this definition says, in essence, that two 2D vectors have the same direction if they are parallel (same slope or no slope for both) and, assuming that they are, they have the same orientation if they both go up or go both down. Notice also how I have excluded the zero vector from this discussion: this will avoid some picky technical questions that we shall revisit at a more useful time.

Can we do the same in 3 dimensions?

We can, but now we don't have a single slope to deal with, so the conditions we used in the last definition become more involved. And that is why we shall soon abandon this approach. But, to see what the problem is and convince you that we need a different approach, let's give it a try:

Definition

Two non-zero 3D vectors $\mathbf{v}=[v_1\ \ v_2\ \ v_3]$ and $\mathbf{w}=[w_1\ \ w_2\ \ w_3]$ have the same ***direction*** if one of the following sets of conditions is satisfied:

1. $v_1\neq0,\ w_1\neq0,\ \frac{v_2}{v_1}=\frac{w_2}{w_1}$ and $\frac{v_3}{v_1}=\frac{w_3}{w_1}$, or
2. $v_2\neq0,\ w_2\neq0,\ \frac{v_1}{v_2}=\frac{w_1}{w_2}$ and $\frac{v_3}{v_2}=\frac{w_3}{w_2}$, or
3. $v_3\neq0,\ w_3\neq0,\ \frac{v_1}{v_3}=\frac{w_1}{w_3}$ and $\frac{v_2}{v_3}=\frac{w_2}{w_3}$.

Two non-zero 3D vectors $\mathbf{v}=[v_1\ \ v_2\ \ v_3]$ and $\mathbf{w}=[w_1\ \ w_2\ \ w_3]$ that have the same direction are said to also have the same ***orientation*** if $v_1 w_1\geq0$, $v_2 w_2\geq0$ and $v_3 w_3\geq0$.

Example:

The vectors $[1\ \ 0\ \ 3]$ and $[2\ \ 0\ \ 6]$ have the same direction, since they satisfy both conditions 1 and 3. They also have the same orientation, since all components are positive.

The vectors $[1\ \ -1\ \ 2]$ and $[-1\ \ 1\ \ -2]$ have the same direction, since they satisfy all three conditions. They have different orientations, since the product of all pairs of corresponding components is negative.

The vectors $[3 \;\; 2 \;\; 5]$ and $[6 \;\; 4 \;\; 0]$ do not have the same direction, since none of the three conditions is satisfied.

Wow! That starts being confusing and difficult to check!

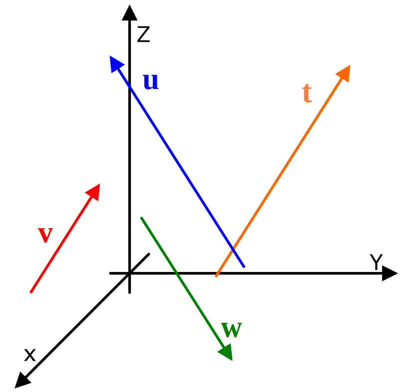
Absolutely, and just imagine how much more confusing it becomes when working in the higher dimensions that we shall see later! That is why we need a different approach, but to do that we also need one technical concept that deserves its own section.

I notice that in the two cases of Example 1.4.4 where the vectors have the same direction, the components are multiples of each other: is that relevant?

Completely! That tells me that you are ready for the next section ☺.

Wait a minute: what about pictures in the 3D case?

They are not easy to draw or to visualize, so I will give you an example here and will try to avoid them in the future for the sake of both of us. Can you see in this picture which vectors have the same direction and/or orientation?



I see what you mean: ***u*** *and* ***w*** *seem to have the same direction, but do they? It is very hard to tell from the picture.*

And again, that is why we shall do very little with 3D pictures it in this course. If you really need to work on visualizing these vectors, you will do it in other courses where this skill is relevant. Here our focus is different and we are not going to sweat on stuff that may distract us from the main goals.

Summary

- The direction of a geometric vector can be defined by using the familiar concept of slope, but it is not an efficient method in 3D.

Common errors to avoid

- Remember that for 3D vectors there are three slopes to consider (or two, by being technical), NOT just one.

Learning questions for Section LA 1-4

Review questions:

1. Explain how to use the concept of slope to define the direction of a 2D vector.

Memory questions:

1. What relation must be satisfied in order for two vectors $\mathbf{u} = [u_1 \ \ u_2]$ and $\mathbf{v} = [v_1 \ v_2]$ to have the same direction?

Proof questions:

1. Prove that in order for two non-zero vectors $\mathbf{u} = [u_1 \ u_2]$ and $\mathbf{v} = [v_1 \ v_2]$ that have the same direction to also have the same orientation, it is sufficient that the product of one pair of corresponding components be positive. Also, explain how this is different from the condition required in the definition.

Templated questions:

1. Construct two 2D vectors and check whether they have the same geometric direction.

2. Construct two 3D vectors and check whether they have the same geometric direction.

3. Construct two 2D vectors that have the same direction.

4. Construct two 3D vectors that have the same direction.

What questions do you have for your instructor?