Chapter 1: Geometric vectors — **Section 5**

Scalar multiplication and algebraic direction of a vector

What you need to know already:	What you can learn here:
➤ Definition of a geometric vectors. ➤ Length and geometric direction of a vector.	➤ The operation of scalar multiplication. ➤ How to use this operation to construct an efficient definition of direction for a vector.

According to our definitions, a 2D vector is simply an ordered pair of usual numbers and a 3D vector is an ordered triple of numbers. Although in each case we can devise and use a geometric representation, the algebraic aspect is the key feature that will be exploited in the rest of the course.

As a way of introduction, I will now show you how, by using the arithmetic operations that are so familiar to us from grade school, we can develop a useful operation for vectors that will lead to a much better way to identify a direction.

Definition

The ***scalar multiplication*** of a scalar c and a vector $\mathbf{v}$ is performed by multiplying each of the components of $\mathbf{v}$ by c. Therefore:

- In 2D: $c\mathbf{v} = c\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} cv_1 & cv_2 \end{bmatrix}$
- In 3D: $c\mathbf{v} = c\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} cv_1 & cv_2 & cv_3 \end{bmatrix}$

Definition

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are ***scalar multiples*** of each other if there is a scalar c such that either $\mathbf{v} = c\mathbf{u}$ or $\mathbf{u} = c\mathbf{v}$.

Example:

By this definition:

$$3\begin{bmatrix} -2 & 4 \end{bmatrix} = \begin{bmatrix} -6 & 12 \end{bmatrix} \text{ and } -2\begin{bmatrix} \pi & 5 \end{bmatrix} = \begin{bmatrix} -2\pi & -10 \end{bmatrix}$$

It turns out that this is an extremely useful operation, one that forms the backbone for most of the material of this book. To start showing you how useful this is, here is the better definition of direction that I have promised:

Technical fact

Two non-zero geometric vectors have the ***same direction*** if and only if they are scalar multiples of each other. Moreover, they have the ***same orientation*** if such scalars are positive.

Proof

Same direction $\Rightarrow$ *Scalar multiples:*

In 2D, if two vectors $\mathbf{v} = [v_1 \;\; v_2]$ and $\mathbf{w} = [w_1 \;\; w_2]$ have the same direction, it means that either they are both vertical, or they have the same slope.

If they are both vertical, they are of the form $\mathbf{v} = [0 \;\; v_2]$ and $\mathbf{w} = [0 \;\; w_2]$ with both v_2 and w_2 different from 0. In that case, if we pick $c = \dfrac{w_2}{v_2}$ we see that the two vectors are scalar multiples of each other.

Scalar multiples $\Rightarrow$ *Same direction*:

If $\mathbf{v} = [v_1 \;\; v_2]$, $\mathbf{w} = [w_1 \;\; w_2]$ and $\mathbf{w} = c\mathbf{v}$, it means that $\mathbf{w} = [cv_1 \;\; cv_2]$ and hence $\dfrac{w_2}{w_1} = \dfrac{cv_2}{cv_1} = \dfrac{v_2}{v_1}$, so that **v** and **w** have the same direction. This argument only fails if $v_1 = 0$, but in that case $w_1 = 0$ as well and hence both vectors are vertical and so they have the same direction.

I leave to you the proof for the 3D case.

You want me to prove it in the more difficult situation? That's rich!

And why do I need to be able to do proofs anyway? I trust what you say ☺

I am honored by your trust, but learning how to do simple proofs is one of the goals of this course, as well as a skill that will be very useful to you in the future. So we all expect you to learn how to do it. As for the difficulty level, in 3D the proof is the same, just longer, since there are more proportions to juggle, but the procedure is the same, so you'll get a good workout.

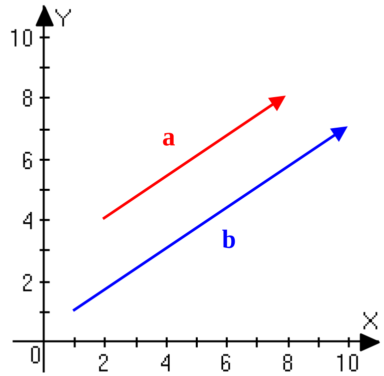
In any case, now that we have a more workable definition of direction, we can look at another word with which you are familiar and that we can now define formally:

Definition

Two geometric vectors are ***parallel*** if they have the same direction, or, equivalently, if they are scalar multiples of each other.

Example:

If we consider the vectors $\mathbf{a} = [6 \;\; 4]$ and $\mathbf{b} = [9 \;\; 6]$, we notice that $\mathbf{b} = 1.5\mathbf{a}$, or $\mathbf{a} = \frac{2}{3}\mathbf{b}$. This means that they are multiples of each other and hence they have the same direction, or are parallel, as the picture clearly shows.



It looks like two vectors are parallel if one of them is a stretched version of the other.

That is correct, since we are maintaining the direction. But there is more!

If you look at the picture you may get the impression that **b** is about 1.5 times longer than **a**. This may seem obvious, but remember that the scalar multiplication of the definition occurs on the components, not the length, so we need to check if this observation is in fact true and not a coincidence:

Technical fact

Scalar multiplication has the effect of ***changing the length*** of the vector by a factor equal to the absolute value of the scalar factor:

$$\|c\mathbf{v}\| = |c| \times \|\mathbf{v}\|$$

Proof

Here is how to prove this fact in 2D:

$$\|c\mathbf{v}\| = \|c\,[v_1 \;\; v_2]\| = \|[cv_1 \;\; cv_2]\| = \sqrt{(cv_1)^2 + (cv_2)^2} =$$

$$= \sqrt{c^2(v_1)^2 + c^2(v_2)^2} = \sqrt{c^2\left[(v_1)^2 + (v_2)^2\right]}$$

$$= \sqrt{c^2}\sqrt{(v_1)^2 + (v_2)^2} = |c| \times \|\mathbf{v}\|$$

Again I leave you the fun of proving this in $\mathbb{R}^3$.

So, we can use scalar multiplication to lengthen and shorten vectors at will, correct?

Yes, and in fact this gives us a very useful way to obtain a unit vector in any direction we want.

Technical fact

Given any non-zero vector **v**, the vector $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is denoted by $\mathbf{e}_{\mathbf{v}}$, it is a unit vector and has the same direction and orientation as **v**.

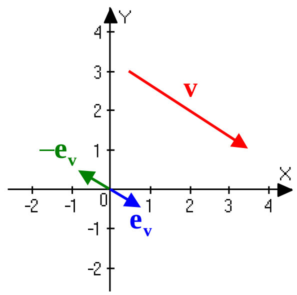
The process of constructing a unit vector in the direction of a given non-zero vector is called ***normalization***.

Example:

If we normalize the vector $\mathbf{v} = [3 \;\; -2]$ we obtain:

$$\mathbf{e}_{\mathbf{v}} = \frac{1}{\sqrt{9+4}}[3 \;\; -2] =$$

$$= \frac{1}{\sqrt{13}}[3 \;\; -2] \approx [0.83 \;\; -0.55]$$



Both vectors are shown in the picture and so is the vector $-\mathbf{e}_{\mathbf{v}} = \frac{1}{\sqrt{13}}[-3 \;\; 2]$, which has the same direction, but opposite orientation

I have a question about jargon and one about notation.

Excellent! Both are very important aspects of this course, so it is best to clarify them as soon as they arise.

First of all, we called this operation "scalar multiplication". Why not the shorter "scalar product?"

It is fine to refer to this operation as the scalar product of a scalar and a vector, but later we shall meet another operation that is also called scalar product, but refers to a rather different situation and produces a different result. So, we can use this terminology as long as the context is clear.

Fair enough. Now, when we multiply a vector by a fraction, can we make the notation smaller by placing the vector in the numerator?

Absolutely! This is a very convenient move and we might as well formalize it in a box.

Definition

A ***scalar division*** of a vector by a scalar is obtained by multiplying the vector by the reciprocal of the scalar. That is:

$$\frac{\mathbf{v}}{c} = \frac{1}{c}\mathbf{v}$$

Before we close this chapter and move to bigger and better things, I want to clarify a little pedantic point that may bother us later if we don't.

When defining what we mean by two vectors being multiples of each other, I excluded consideration of the zero vector, for otherwise we may have generated a division by 0 in the formula. That is also reasonable because the zero vector does not have a proper direction (it includes a single point) and because the constant that would change any vector to the zero vector is 0, but one cannot divide by zero in order to reverse the relation.

Although reasonable, it turns out that this exclusion would give us lots of headaches in future developments. To avoid this problem, we shall use the following convention:

Definition

The zero vector is considered to be a ***multiple*** of any other vector, hence ***parallel*** to any other vector, despite the fact that it does not have a direction and that the algebraic relationship of being scalar multiple would only be satisfied in one direction.

I am delaying examples of this definition because at this point their simplicity would offend your intelligence and because we shall have many opportunities to see it in action when it counts. For now, just keep it in mind and write down any questions you may have about it.

Summary

- Since scalar multiplication changes each component by the same factor, it does not affect the direction of a vector. Therefore, it can be used to identify such direction through an equivalence relation.
- Two vectors have the same direction, or are parallel, if they are multiples of each other.
- A unit vector can be seen as being representative of a direction.

Common errors to avoid

- Distinguish between direction and orientation: they are related concepts, but not the same thing!
- Remember that the zero vector is considered as being parallel to any other vector, even though this slightly violates the mutual requirement of the definition.

Learning questions for Section LA 1-5

Review questions:

1. Describe how scalar multiplication works.
2. Explain how scalar multiplication is used to define the direction of a geometric vector.
3. Discuss the role of unit vectors in the concept of algebraic direction of a vector.

Memory questions:

1. Which formula defines the product of a scalar and a 3D vector?
2. Which formula defines the unit vector with the same direction and orientation of a given vector **v**?

Computation questions:

1. Given the vectors $\mathbf{v} = \begin{bmatrix} 3 & -5 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} \pi & 5.2 & \sqrt{3} \end{bmatrix}$, determine 3**v** and -2**u**.
2. Determine the unit vectors parallel to $\mathbf{v} = \begin{bmatrix} 3 & -5 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} \pi & 5.2 & \sqrt{3} \end{bmatrix}$.
3. Which vector is the unit vector in the direction of the vector **PQ** joining **P** $(-3, 2)$ and **Q** $(-1, -2)$?
4. Determine the unit vector of **v** that has the opposite orientation of the vector $\mathbf{v} = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}$.
5. Determine the magnitude and the direction – as a unit vector – of the vector whose tail is at the point $(3, 1, 0)$ and whose tip is at $(1, -2, 4)$.

6. Find all values of *k* for which the vectors $\mathbf{u} = \begin{bmatrix} k & k & 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 & k & 1 \end{bmatrix}$ are parallel.

7. Given the vectors $\mathbf{u} = \begin{bmatrix} 1 & 2 & k \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 & 6 & 9 \end{bmatrix}$, determine the values of *k* for which, respectively:
   a) **u** and **v** are parallel and with the same orientation;
   b) **u** and **v** are parallel and with opposite orientation.

Theory questions:

1. How is a unit vector obtained from a given vector **v**?
2. What is the main purpose of unit vectors?
3. What is the geometric effect of multiplying a vector by a scalar?
4. Given a vector $\mathbf{v} = \begin{bmatrix} a & b & c \end{bmatrix}$, construct a vector **w** that has half its length.
5. How many unit vectors have the same direction as a given vector?
6. Which are the unit vectors in the direction of the line $y = x$?

Proof questions:

1. Show that if two 2D vectors $\begin{bmatrix} a_1 & a_2 \end{bmatrix}$ and $\begin{bmatrix} b_1 & b_2 \end{bmatrix}$ have the same direction and $a_1 \times b_1 > 0$, then they have the same orientation.
2. Prove that scalar multiplication is *distributive*, that is, for any geometric vectors **u** and **v** and any scalars *h* and *k*, $h(\mathbf{u} + \mathbf{v}) = h\mathbf{u} + h\mathbf{v}$ and $(h + k)\mathbf{u} = h\mathbf{u} + k\mathbf{u}$.
3. Prove that scalar multiplication is *associative*, that is, for any geometric vector **u** and scalars *h* and *k*, $h(k\mathbf{u}) = (hk)\mathbf{u}$.
4. Prove that the scalar 1 is an *identity* for scalar multiplication, that is, for any geometric vector **v**, $1\mathbf{v} = \mathbf{v}$.
5. Use vectors to prove that the midpoints of the quadrilateral formed by the points $(-3, -1), (1, -3), (4, 1), (-1, 5)$ is a parallelogram.
6. Use vectors to prove that the midpoints of the quadrilateral formed by any four points in the Cartesian plane is a parallelogram. This is known as *Varignon's theorem*.

Templated questions:

In these questions, make your own choice of any items involved.

1. Choose a 2D vector and then construct the unit vector parallel to it and with the same orientation.
2. Choose a 3D vector and then construct the unit vector parallel to it and with the same orientation.
3. Choose a 2D vector and then construct the unit vector parallel to it and with the opposite orientation.
4. Choose a 3D vector and then construct the unit vector parallel to it and with the opposite orientation.

What questions do you have for your instructor?