Chapter 1: Geometric vectors **Section 6**

Vector addition

What you need to know already:	What you can learn here:
➤ What geometric vectors are. ➤ What is meant by length and direction of a vector. ➤ The scalar product.	➤ How to add two geometric vectors, both algebraically and geometrically.

In regular algebra we know how to add two numbers and even two variables, two polynomials, two functions etcetera. After all addition is a simple operation that we have all learned early in our life. But vectors are just ordered sets of numbers, so we should be able to add them in the same way, just keeping the ordered set format.

Definition

The ***sum*** of two vectors is the vector obtained by adding the corresponding components of the original vectors.

Therefore, if $\mathbf{v} = [v_1 \;\; v_2]$ and $\mathbf{w} = [w_1 \;\; w_2]$ are 2D vectors, then:

$$\mathbf{v} + \mathbf{w} = [v_1 \;\; v_2] + [w_1 \;\; w_2] = [v_1 + w_1 \;\; v_2 + w_2]$$

Similarly, if $\mathbf{v} = [v_1 \;\; v_2 \;\; v_3]$ and $\mathbf{w} = [w_1 \;\; w_2 \;\; w_3]$ are 3D vectors:

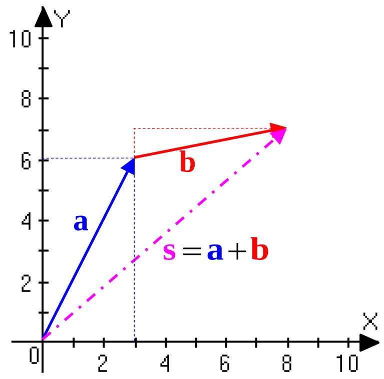
$$\mathbf{v} + \mathbf{w} = [v_1 \;\; v_2 \;\; v_3] + [w_1 \;\; w_2 \;\; w_3] = [v_1 + w_1 \;\; v_2 + w_2 \;\; v_3 + w_3]$$

This definition can be extended in the obvious way to the sum of ***several*** vectors.

I can see that this operation is very simple, but also very boring!

I agree, but it becomes more interesting if we start seeing what effect it has on both the geometry of these vectors and the applications. I will show you in 2 dimensions, where drawing is easy.

Consider the vectors $\mathbf{a} = [3 \;\; 6]$ and $\mathbf{b} = [5 \;\; 1]$, and let us draw the first one starting at the origin and the second starting at the tip of the first, as shown in this picture:



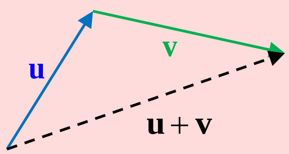
If we now consider the sum of these two vectors:

$$\mathbf{s}=\mathbf{a}+\mathbf{b}=\begin{bmatrix}3 & 6\end{bmatrix}+\begin{bmatrix}5 & 1\end{bmatrix}=\begin{bmatrix}8 & 7\end{bmatrix}$$

and use the representation of **s** that also starts at the origin, you will notice that these three geometric vectors form a triangle.

Technical fact: the triangle rule

If two vectors **u** and **v** are added, a geometrical representation of their sum **u+v** is given by the third side of a ***triangle*** whose other two sides are represented by **u** and **v**, with the tail of **v** being at the tip of **u**.

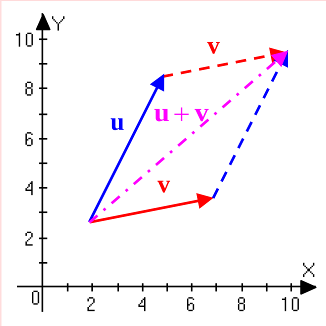


You may want to use some elementary analytic geometry to prove this simple fact, or, to simply convince yourself that this is not a coincidence, try the same procedure on some other pairs of vectors and check that in this way you always obtain a triangle.

The name *"triangle rule"* is not used often, since there is another geometrical way to visualize the sum of two vectors that works in a similar way, but looks at it from a different perspective.

Technical fact: the parallelogram rule

If two vectors **u** and **v** are added, a geometrical representation of their sum **u+v** is the diagonal of the ***parallelogram*** whose two sides with common vertex are represented by **u** and **v**, with the tail of both being at the same point.



I guess the parallelogram rule must have used a better public relations firm!

In reality this approach is more popular because it is very effective when vectors represent forces acting on an object. In that situation we think of such forces as being applied to the same point, rather than back to back.

Obviously the two representations are equivalent, but the triangle rule has several nice generalizations (a staple of linear algebra) that are not so clear when addition is seen from the perspective of the parallelogram rule. For instance:

Technical fact: the triangle inequality

For any two geometric vectors **u** and **v**:

$$\|\mathbf{u}+\mathbf{v}\|\le\|\mathbf{u}\|+\|\mathbf{v}\|.$$

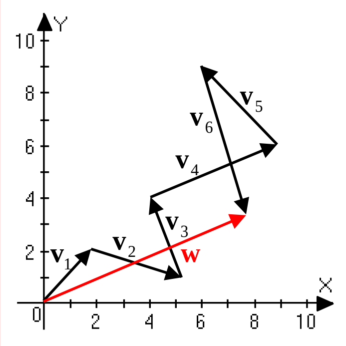
That's because any side of a triangle is shorter than the sum of the other two!

Exactly! Here is another:

Technical fact: the polygonal rule

When we add several vectors, say

$$\mathbf{w} = \mathbf{v}_1 + \mathbf{v}_2 + \ldots + \mathbf{v}_k$$



and represent each of them with an arrow so that the first one starts at any point and each of the other arrows starts where the previous one ends, their sum is represented by an arrow that starts at the tail of the first arrow and ends at the tip of the last one, as shown in this picture.

This name is also not used as commonly as the parallelogram rule, although it is an interesting one: even in math things aren't always fair ☹.

Summary

- Two 2D vectors or two 3D vectors can be added by adding the corresponding components.
- The sum of two geometric vectors can be represented geometrically by using the triangle rule.

Common errors to avoid

- Remember that the arrows we use in the triangle rule are NOT the vectors! They are graphical representations of the vectors, with the rule being a useful graphical tool to see the effect of such addition.

Learning questions for Section LA 1-6

Review questions:

1. Describe how two geometric vectors are added.

2. Explain how the triangle, parallelogram and polygonal rules provide a graphical model for vector addition.

Memory questions:

1. Which geometrical rules are associated with vector addition?

Computation questions:

1. Find the other three vertices of the parallelogram whose first vertex is at the point $P_1 = (1, 2, 1)$ and whose sides are parallel to the vectors $\mathbf{u} = \begin{bmatrix} 2 & -2 & 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 & 1 & -2 \end{bmatrix}$.

2. Find the lengths of the two diagonals of the parallelogram whose first vertex is at the point $P = (1, 2, 1)$ and whose sides are parallel to the vectors $\mathbf{u} = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 & 0 & -2 \end{bmatrix}$.

Proof questions:

1. Prove that vector addition is *commutative*, that is, prove that for any geometric vectors **u** and **v**, it is true that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

2. Explain why this commutative property also extends to the polygonal rule.

3. Prove that vector addition is *associative*, that is, prove that for any geometric vectors **u**, **v**,and **w**, it is true that $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

4. Prove that the zero vector is *neutral* for addition, meaning that for any geometric vector **v**, $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$

5. Prove that any geometric vector **v** has an *opposite*, that is, there is another vector **w** such that $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} = \mathbf{0}$.

6. The difference $\mathbf{u}-\mathbf{v}$ between two vectors can be easily defined exactly in the same way as for addition, that is, component-wise. But what is the geometrical interpretation, in 2D, of such a difference?

7. Prove the polygonal rule.

Application questions:

1. When two vectors are used to represent forces applied to the same point, their sum, also called their *resultant*, is obtained through the parallelogram rule. Compute the resultant of two forces acting on the same object, one with strength 30N directed towards North and the other with strength 45N directed towards East.

Templated questions:

1. Generate several 2D vectors and add them up, both algebraically and geometrically.

2. Generate several 3D vectors and add them up algebraically.

What questions do you have for your instructor?