Chapter 1: Geometric vectors *Section 8*

The dot product

What you need to know already:	*What you can learn here:*
➤ What a linear combination of vectors is.	➤ How to use two vectors to obtain a scalar that contains important information about the two vectors.

So far we have seen how to add vectors and how to multiply a vector by a scalar, two operations that combine to generate linear combinations. But there is also a very useful way of multiplying two vectors together, useful in that it will allow us to perform all kinds of computations and explore all kinds of interesting concepts and applications. It is a strange product, though, in the sense that it produces something that is different from both of the two items that we multiply together.

Definition

The ***scalar product***, or ***dot product*** of two 2D vectors **u** and **v** is denoted by $\mathbf{u} \bullet \mathbf{v}$ and is computed as the sum of the products of the components in corresponding positions. In formula:

$$\mathbf{u} \bullet \mathbf{v} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \bullet \begin{bmatrix} v_1 & v_2 \end{bmatrix} = u_1 v_1 + u_2 v_2$$

Of course the idea can be easily extended to 3D vectors.

Definition

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$$\mathbf{u} \bullet \mathbf{v} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \bullet \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Knot on your finger

Notice that the dot product of any two vectors is always a ***scalar***.

Is it called a scalar product because it creates a scalar from two vectors?

Yes, although I will most often use the expression *dot product*, so as to avoid confusion with the scalar multiplication, which combines not two vectors, but a scalar and a vector. (You may remember that I pointed to this issue in Section 1.6)

And what is the dot product good for?

The dot product has many interesting and useful applications that in a short time will become second nature to you, if you don't know them already from other courses. But watch out, because soon we shall investigate not only the dot product, but its very properties. So, pay attention not only to the mechanical side of the formulae, but also to their meaning and workings.

Here are some simple and familiar algebraic properties to get us started:

Technical fact:

Algebraic properties of the dot product

The dot product is ***commutative***, meaning that if we multiply the two vectors in reverse order we obtain the same result:

$$\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$$

The dot product is ***distributive*** *with respect to vector addition*, meaning that the *FOIL* process applies to it:

$$(\mathbf{a}+\mathbf{b}) \bullet (\mathbf{u}+\mathbf{v}) = \mathbf{a} \bullet \mathbf{u} + \mathbf{a} \bullet \mathbf{v} + \mathbf{b} \bullet \mathbf{u} + \mathbf{b} \bullet \mathbf{v}$$

The dot product is ***associative with respect to scalar multiplication***, meaning that a scalar multiple can be attached to any factor of the product:

$$c(\mathbf{u} \bullet \mathbf{v}) = (c\mathbf{u}) \bullet \mathbf{v} = \mathbf{u} \bullet (c\mathbf{v})$$

The dot product is ***positive definite***, meaning that if $\mathbf{v}$ is any non-zero vector, then $\mathbf{v} \bullet \mathbf{v} > 0$, while $\mathbf{0} \bullet \mathbf{0} = 0$.

Proof

Commutativity is a consequence of the fact that the dot product is based on the usual addition and multiplication, both of which are commutative.

I will leave to you the proof of the ***distributivity*** with respect to vector addition, which goes along the same lines as the other one, coming up…

To prove that the dot product is ***associative*** with respect to scalar multiplication, all we need to do is look at what the three combinations of operations presented do on the components. I will do so in 2D, since the 3D case is exactly the same, just longer:

$$c(\mathbf{u} \bullet \mathbf{v}) = c(u_1v_1 + u_2v_2) = cu_1v_1 + cu_2v_2$$
$$(c\mathbf{u}) \bullet \mathbf{v} = [cu_1 \;\; cu_2] \bullet [v_1 \;\; v_2] = cu_1v_1 + cu_2v_2$$
$$\mathbf{u} \bullet (c\mathbf{v}) = [u_1 \;\; u_2] \bullet [cv_1 \;\; cv_2] = u_1cv_1 + u_2cv_2$$

and since the usual product of scalars is commutative, all three combinations do provide the same thing.

The proof of the ***positive definite*** property follows immediately from the definition and you may want to check it as an exercise.

Snore, snore snore…

I suggest you wake up, as there may be some tricky details to consider, such as the following.

Knot on your finger

The dot product is ***associative*** with respect to the scalar multiplication, but it is NOT associative by itself.

What do you mean "by itself"?

Associativity is usually a property of a single operation and it refers to how we can combine three objects by using the operation twice. For the dot product that does not work, as you will discover in the *Checkpoint*.

But I must agree that this algebraic gymnastic is rather boring unless math is your hobby or profession. These properties will become more relevant later, but let us now look at some very simple and useful geometric properties whose value should be rather clear.

Technical fact:

Geometric properties of the dot product

The ***length*** of any geometric vector **v** is given by:

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \bullet \mathbf{v}}$$

If **u** and **v** are any two geometric vectors and θ is the ***angle*** between them, that is, the smaller angle between two of their representative arrows with the same tail, then:

$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

If **u** and **v** are any two geometric vectors and θ is the ***angle*** between them, then:

$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

Two geometric vectors **u** and **v** are ***orthogonal*** (that is, ***perpendicular***) if and only if their dot product is 0.

Before I give you the formal proof of these statements, let us see how they work in some familiar cases.

Example:

The vectors $\begin{bmatrix}2 & 0\end{bmatrix}$ and $\begin{bmatrix}3 & 3\end{bmatrix}$ form a 45º angle (check it thou out!), so the ratio that is purported to equal the cosine of the angle should equal $\cos 45^\circ$. Let's see:

$$\frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{[2\ \ 0]\bullet[3\ \ 3]}{\sqrt{[2\ \ 0]\bullet[2\ \ 0]}\sqrt{[3\ \ 3]\bullet[3\ \ 3]}} = \frac{6}{\sqrt{4}\sqrt{18}} = \frac{6}{2\sqrt{2\times 9}} = \frac{1}{\sqrt{2}}$$

Yep! It works!

The vectors $\begin{bmatrix}1 & 0 & 3\end{bmatrix}$ and $\begin{bmatrix}0 & 4 & 0\end{bmatrix}$ form a 90º angle (check it out!), so the ratio that is purported to equal the cosine of the angle should equal cos90º, which is 0. Let's see:

$$\frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{[1\ \ 0\ \ 3]\bullet[0\ \ 4\ \ 0]}{\sqrt{[1\ \ 0\ \ 3]\bullet[1\ \ 0\ \ 3]}\sqrt{[0\ \ 4\ \ 0]\bullet[0\ \ 4\ \ 0]}} = \frac{0}{\sqrt{10}\sqrt{16}} = 0$$

It works again!

Hopefully these examples have already given you a glimpse of why the statements are true in general. But remember that a big part of linear algebra is checking proofs, and this is one situation where the needed proofs are simple enough that you can follow them, but require enough different ideas and are general enough that your undivided attention is needed.

Proof

Length:

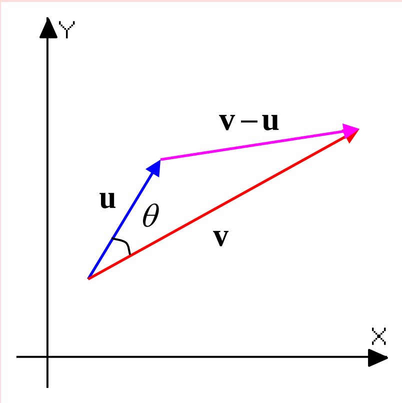
First we need to show that $\|\mathbf{v}\| = \sqrt{\mathbf{v} \bullet \mathbf{v}}$ for any vector **v**. This is done simply by checking:

$$\|\mathbf{v}\| = \left\|\begin{bmatrix}v_1 & v_2\end{bmatrix}\right\| = \sqrt{v_1^2 + v_2^2} = \sqrt{v_1v_1 + v_2v_2} = \sqrt{\mathbf{v} \bullet \mathbf{v}}$$

Angle:

To show that $\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$ we shall use the law of cosines, which states that if *a*, *b* and *c* are the three sides of any triangle and θ is the angle between *a* and *b*, then $c^2 = a^2 + b^2 - 2ab\cos\theta$.

Now let us use representative arrows of **u** and **v** starting at the same point and consider the representative arrow for **v-u** that starts at the tip of **u**.



Because of the triangle rule, this configuration creates a triangle, as shown in the figure. We now apply the law of cosines to it and obtain:

$$\|\mathbf{v}-\mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

But the previous property of the dot product implies that the square of the length of a vector equals the dot product of the vector with itself. Therefore:

$$(\mathbf{v}-\mathbf{u})\bullet(\mathbf{v}-\mathbf{u}) = \mathbf{u}\bullet\mathbf{u} + \mathbf{v}\bullet\mathbf{v} - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

We can now use the algebraic properties of the dot product to change the appearance of the left side:

By distributivity:

$$(\mathbf{v}-\mathbf{u})\bullet(\mathbf{v}-\mathbf{u}) = \mathbf{v}\bullet\mathbf{v} + \mathbf{v}\bullet(-\mathbf{u}) + (-\mathbf{u})\bullet\mathbf{v} + (-\mathbf{u})\bullet(-\mathbf{u})$$

By associativity:

$$= \mathbf{v}\bullet\mathbf{v} - \mathbf{v}\bullet\mathbf{u} - \mathbf{u}\bullet\mathbf{v} + \mathbf{u}\bullet\mathbf{u}$$

By commutativity:

$$= \mathbf{v}\bullet\mathbf{v} - \mathbf{u}\bullet\mathbf{v} - \mathbf{u}\bullet\mathbf{v} + \mathbf{u}\bullet\mathbf{u}$$

And by regular algebra of scalars:

$$= \mathbf{v}\bullet\mathbf{v} - 2\mathbf{u}\bullet\mathbf{v} + \mathbf{u}\bullet\mathbf{u}$$

We are almost there! If we now use this expression as the left side of the law of cosines equation, we obtain:

$$\mathbf{v}\bullet\mathbf{v} - 2\mathbf{u}\bullet\mathbf{v} + \mathbf{u}\bullet\mathbf{u} = \mathbf{u}\bullet\mathbf{u} + \mathbf{v}\bullet\mathbf{v} - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

And by cancelling the two pairs of like terms we obtain:

$$\Rightarrow\; -2\mathbf{u}\bullet\mathbf{v} = -2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta \;\Rightarrow\; \mathbf{u}\bullet\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

Just the property that we needed to prove!

Notice that although the picture that I used suggests a proof in 2D, exactly the same proof works in 3D, since the cosine law applies to any triangle in any plane!

If we move the product of the two lengths in the last equation to the left side, we obtain the formula for the cosine of the angle between two vectors (and hence the angle itself). Notice that this formula generates the cosine by using only the components of the two vectors: trigonometry by using only addition and multiplication! Ain't it neat?

Orthogonality:

Two geometric vectors are perpendicular if and only if the angle between them is $\pi/2$ (90° if you prefer, since here we are dealing with static angles). But the cosine of this angle is 0, hence the property.

Do you see now why I presented the algebraic properties of the dot product first?

Because you needed them to prove the geometric properties!

Yes, and this is another example of how algebra, boring as it may seem, is behind many useful applications. Too bad that people see and enjoy the applications, but refuse to see the relevance of the tools that make them possible. But I am digressing, let's see two more examples.

Example:

The vectors $\begin{bmatrix}3 & 4 & 0\end{bmatrix}$ and $\begin{bmatrix}9 & 12 & 5\sqrt{3}\end{bmatrix}$ form a 30º angle. It is possible to prove this by using the fact that the second vector is straight above the first and its length is twice its third component, but that is somewhat involved (see if you can do it, though). But if we use the relation between dot product and cosine we get:

$$\cos\theta = \frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\begin{bmatrix}3 & 4 & 0\end{bmatrix}\bullet\begin{bmatrix}9 & 12 & 5\sqrt{3}\end{bmatrix}}{\sqrt{\begin{bmatrix}3 & 4 & 0\end{bmatrix}\bullet\begin{bmatrix}3 & 4 & 0\end{bmatrix}}\sqrt{\begin{bmatrix}9 & 12 & 5\sqrt{3}\end{bmatrix}\bullet\begin{bmatrix}9 & 12 & 5\sqrt{3}\end{bmatrix}}}$$

$$= \frac{27+48}{\sqrt{9+16}\sqrt{81+144+75}} = \frac{75}{\sqrt{25}\sqrt{300}} = \frac{75}{5\times 10\sqrt{3}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

But if the cosine has this value, the angle must be 30º. Very simple eh?

Example:

What is the angle between $\begin{bmatrix}-2 & 7 & 4\end{bmatrix}$ and $\begin{bmatrix}2 & -1 & 5\end{bmatrix}$? Figuring this one out geometrically is mind-boggling, but look:

$$\cos\theta = \frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\begin{bmatrix}-2 & 7 & 4\end{bmatrix}\bullet\begin{bmatrix}2 & -1 & 5\end{bmatrix}}{\sqrt{\begin{bmatrix}-2 & 7 & 4\end{bmatrix}\bullet\begin{bmatrix}-2 & 7 & 4\end{bmatrix}}\sqrt{\begin{bmatrix}2 & -1 & 5\end{bmatrix}\bullet\begin{bmatrix}2 & -1 & 5\end{bmatrix}}}$$

$$= \frac{-4-7+20}{\sqrt{69}\sqrt{30}} = \frac{9}{\sqrt{2070}}$$

If we now want the angle, we use our friendly calculator to obtain:

$$\theta = \cos^{-1}\frac{9}{\sqrt{2070}} \approx 78.6^\circ$$

And after seeing something very effective, let me show you something really weird. Do you remember that we agreed that the zero vector is parallel to any vector? Well, look at this:

Technical fact

The zero vector is ***perpendicular*** to any vector, both in 2D and 3D.

The proof of this fact is practically obvious, but notice that it tells us that the zero vector is a very special vector, since it is both parallel and perpendicular to any vector. This will pop up again in the future.

Summary

- The dot product uses a simple arithmetic formula that only involves the components of the two vectors involved.
- The dot product can provide useful information about both lengths and angles of the vectors involved.
- According to our definitions, the zero vector is both parallel and perpendicular to any other vector.

Common errors to avoid

- The formulae that provide lengths and angles in terms of the dot product are simple, but not trivial: make sure to memorize and use them properly.
- You may think that the proofs of the key properties shown in the section need not be understood. Think again! They contain important steps and properties with which you need to become familiar, so do not underestimate their importance and make sure you do understand them.

Learning questions for Section LA 1-8

Review questions:

1. Describe how to perform the dot product of two vectors.
2. Explain how the dot product provides information about the length of a vector.
3. Explain how the dot product provides information about the angle between two vectors.

Memory questions:

1. Which formula defines the dot product for 2D vectors?
2. Which formula defines the dot product for 3D vectors?
3. Which formula provides the connection between dot product and the length of a vector?
4. Which formula provides the connection between dot product and the angle between two vectors?
5. What is the dot product of two perpendicular vectors?

Computation questions:

Compute the dot products presented in questions 1-4.

1. $\begin{bmatrix}3 & -2\end{bmatrix} \bullet \begin{bmatrix}-4 & 3\end{bmatrix}$

2. $\begin{bmatrix}2 & 3\end{bmatrix} \bullet \begin{bmatrix}-4 & 3\end{bmatrix}$

3. $\begin{bmatrix}1 & 0 & -2\end{bmatrix} \bullet \begin{bmatrix}-3 & 4 & 1\end{bmatrix}$

4. $\begin{bmatrix}2 & 1 & -2\end{bmatrix} \bullet \begin{bmatrix}-3 & 4 & 1\end{bmatrix}$

5. Determine whether the vectors $\begin{bmatrix}-1 & 5 & 7\end{bmatrix}$ and $\begin{bmatrix}-4 & 2 & -2\end{bmatrix}$ are perpendicular to each other.

6. Determine whether the vectors $\begin{bmatrix}-1 & 7 & 5\end{bmatrix}$ and $\begin{bmatrix}-2 & 4 & -2\end{bmatrix}$ are perpendicular to each other.

7. Find all values of *k* for which the vectors $\mathbf{u} = \begin{bmatrix}k & k & 2\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}3 & k & 1\end{bmatrix}$ are parallel and those for which they are perpendicular, if any!

8. Determine the value(s) of *k* for which the vectors $\begin{bmatrix}a & k & b\end{bmatrix}$, $\begin{bmatrix}c & k & k\end{bmatrix}$ are parallel and those for which they are perpendicular, if any!

9. Find all values of *k* (if any) for which the vectors $\mathbf{u} = \begin{bmatrix}-3 & 1 & 2\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}2 & k & 1\end{bmatrix}$ form an angle of:
   a) π/4 b) π/2

10. Find all values of *k* (if any) for which the vectors $\mathbf{u} = \begin{bmatrix}k & 4 & 2\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}3 & k & 1\end{bmatrix}$ form an angle of:
    a) π/4 b) π/2

11. Find all values of *k* (if any) for which the vectors $\mathbf{u} = \begin{bmatrix}k & 4 & 2\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}\sqrt{10} & k & \sqrt{10}\end{bmatrix}$ form an angle of:
    a) π/3 b) π/2

12. Find all values of *k*, if any, for which the vectors $\mathbf{u} = \begin{bmatrix}k^2 & k & 2\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}3 & k & -1\end{bmatrix}$ are:
    a) at a π/4 angle b) orthogonal

13. Given the vectors $\mathbf{u} = \begin{bmatrix}1 & 2 & k\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}3 & 6 & 9\end{bmatrix}$, determine the values of *k* for which, respectively:
    a) **u** and **v** are parallel;
    b) **u** and **v** are perpendicular;
    c) the projection of **u** on **v** has length 1.

14. Given that $\mathbf{u} = \begin{bmatrix}1 & 3\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}-2 & 1\end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix}2 & -3\end{bmatrix}$, determine if the vector $2(\mathbf{u}-\mathbf{v}) + \mathbf{v} \bullet \mathbf{w}(\mathbf{u}+\mathbf{w})$ is perpendicular to **w**.

15. Determine whether the vectors $\mathbf{u} = \begin{bmatrix}3 & -\sqrt{2} & 5\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}2 & \sqrt{8} & -3\end{bmatrix}$ form an acute or an obtuse.

16. Construct a linear combination of the vectors $\mathbf{u} = \begin{bmatrix}1 & 2 & 3\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}3 & -1 & 2\end{bmatrix}$ that is orthogonal to $\mathbf{w} = \begin{bmatrix}-2 & 3 & 1\end{bmatrix}$ and has length $\sqrt{21}$, or explain why such a vector does not exist.

17. Construct a vector perpendicular to $\begin{bmatrix}1 & 2 & -3\end{bmatrix}$.

18. Construct a vector perpendicular to $\begin{bmatrix}-2 & 3 & \pi\end{bmatrix}$.

In questions 19-20, determine whether the two given vectors form an acute or an obtuse angle.

19. $\mathbf{u} = [3, 2, -2]$ and $\mathbf{v} = [-1, 2, 5]$

20. $\mathbf{u} = [3, -\sqrt{2}, 5]$ and $\mathbf{v} = [2, \sqrt{3}, -3]$

21. Given the point $P = (2, 4, -1)$ and the vector $\mathbf{v} = [6 \ \ -1 \ \ -2]$, find the tip of the arrow whose tail is at P and that represents, respectively:
  a) a non-zero vector parallel to v;
  b) a non-zero vector orthogonal to **v**.

22. Given the points $P = (2, 4, -1)$ and $Q = (6, -1, -2)$, find:
  a) two different unit vectors parallel to $\mathbf{v} = \overline{PQ}$;
  b) two non-parallel vectors both orthogonal to $\mathbf{v} = \overline{PQ}$.

Theory questions:

Determine which of the expressions involving vectors and scalar and presented in questions 1-4 are meaningful and which ones are not. Provide a suitable rationale in each case.

1. $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

2. $(\mathbf{u} + \mathbf{v}) \bullet (\mathbf{u} \bullet \mathbf{w})$

3. $k(\mathbf{u} \bullet \mathbf{v}) + \mathbf{w}$

4. $k(\mathbf{u}) + \dfrac{\mathbf{u} + \mathbf{v}}{\mathbf{w} \bullet \mathbf{v}}$

5. Which vector operation assigns to each pair of vectors a scalar?

6. Which linear algebra relation is satisfied by orthogonal vectors?

7. Provide one vector orthogonal to **i**

8. If **u** and **v** are vectors of the same dimension, is the quantity $5(\mathbf{u} \bullet \mathbf{v})(\mathbf{u} - \mathbf{v})$ a scalar, a vector, or undefined?

9. Why is the expression $\mathbf{a} \bullet \mathbf{b} + \mathbf{c}$ meaningless?

10. What geometrical information can be obtained through the dot product?

Proof questions:

1. Prove that the dot product is distributive with respect to vector addition.

2. Check that the dot product is positive definite.

3. Prove that any vector orthogonal to $\mathbf{v} = [v_1 \ \ v_2]$ is of the form $\mathbf{n}_\mathbf{v} = [-kv_2 \ \ kv_1]$.

4. Develop a method to construct a vector perpendicular to a given vector $\mathbf{v} = [v_1 \ \ v_2]$ in 2D.

5. Develop a method to construct a vector perpendicular to a given vector $\mathbf{v} = [v_1 \ \ v_2 \ \ v_3]$ in 3D. Notice that this is not exactly the same as for the previous question, since a little extra step is required.

6. Use the dot product and its properties (that is, no geometry) to prove that if **u** and **v** are orthogonal geometric vectors, $\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ and $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$. Do you recognize these two statements as a very famous geometrical fact?

7. Explain why two vectors **u** and **v** form an acute angle if the dot product $\mathbf{u}\bullet\mathbf{v}$ is positive and an obtuse angle if such dot product is negative.

8. Explain why the dot product is not associative.

9. Given a vector $\mathbf{v} = [a \ \ b \ \ c]$, construct a vector **w** that is perpendicular to it and has half its length.

10. Show that there are infinitely many vectors **w** orthogonal to a given 3D vector **v**.

Application questions:

1. If **u** and **v** are orthogonal vectors, write the Pythagorean identities that link them in terms of dot products.

2. Use the notation and properties of dot products and of vector magnitudes to show that the points $\mathbf{P}(-3, 2)$, $\mathbf{Q}(-1, -2)$ and $\mathbf{R}(1,-1)$ form a right triangle by showing that:
   a) two of their sides are perpendicular
   b) the three sides satisfy the Pythagorean theorem.

3. Use the dot product and the parallelogram rule for the addition of vectors to prove that in a parallelogram, the sum of the squares of the two diagonals equals the sum of the squares of all four sides.

4. Obtain a proof of the same statement given in the previous question, but without using linear algebra methods, that is, by using basic plane geometry and trigonometry. Which is easier?

Templated questions:

1. Pick any two vectors and compute their dot product.

2. Pick any two vectors and compute the angle they form.

What questions do you have for your instructor?