Chapter 1: Geometric vectors | **Section 9**

Orthogonal projections

What you need to know already:	*What you can learn here:*
➤ The formula and meaning of the dot product.	➤ How to obtain the geometrical construction of the projection by using linear algebra methods.

The geometric properties of the dot product are due to its connection to the angle between the two vectors. The application you are about to see also uses dot product and the angle between the vectors, but to get rid of that angle – in some sense! Let's start from a familiar definition.

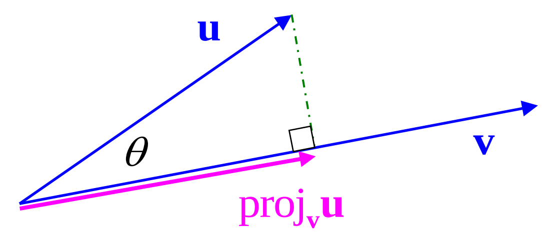
Definition

The ***orthogonal projection*** (or simply ***projection***) of a vector **u** on a vector **v** is given by the vector $\text{proj}_{\mathbf{v}}\mathbf{u}$ that is parallel to **v** and whose representative arrow with the same tail as **u** forms one side of a right triangle whose hypotenuse is **u**.

Do you expect me to understand this smorgasbord of words?

Yes, although you may need to do some parsing, but that is a feature of linear algebra, remember?

To make your task easier, here is a geometrical illustration of this concept. I am confident that it fits with your intuitive idea of what a projection is.



By the way, if you can come up with a clearer, but still formally correct definition, please let me know!

I'll get back to you on that…

In the meantime, I will show you how we can use the properties of vectors and dot products that we know so far to get a simple formula to determine this projection, a formula that can even be used as its definition!

Technical fact

The ***projection*** of a geometric vector **u** on another vector **v** of the same dimension is given by the formulae:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{v}\bullet\mathbf{v}}\mathbf{v}$$

The length of this projection, called the ***scalar projection*** of **u** on **v,** and is given by the formula:

$$\|\text{proj}_{\mathbf{v}}\mathbf{u}\| = \frac{|\mathbf{u}\bullet\mathbf{v}|}{\|\mathbf{v}\|}$$

I hope you will agree with me that this is a simple formula, easy to remember and easy to use.

A little too simple! It almost looks like magic: where does it come from?

Oh, an old book I once read ☺. I suspect that what you mean to ask is *"what is its proof"*, so here it is.

Proof

If you look at the picture of the projection again, you will notice that the vector we are after has the same direction as **v** and its length is $\|\mathbf{u}\|\cos\theta$. By using the unit vector of **v** we can therefore state that:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \|\mathbf{u}\|\cos\theta\frac{\mathbf{v}}{\|\mathbf{v}\|}$$

But now we can use the formula that expresses the cosine in terms of dot products and projections, as well as a little usual algebra:

$$\|\mathbf{u}\|\cos\theta\frac{\mathbf{v}}{\|\mathbf{v}\|} = \|\mathbf{u}\|\frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\frac{\mathbf{v}}{\|\mathbf{v}\|}$$
$$= \frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{v}\bullet\mathbf{v}}\mathbf{v}$$

Hence the formula for the projection. To obtain the scalar projection we need the length of this vector:

$$\left\|\frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}\right\| = \frac{|\mathbf{u}\bullet\mathbf{v}|}{\|\mathbf{v}\|^2}\|\mathbf{v}\| = \frac{|\mathbf{u}\bullet\mathbf{v}|}{\|\mathbf{v}\|}$$

as claimed.

Just a little geometry and the trigonometry is tossed out!

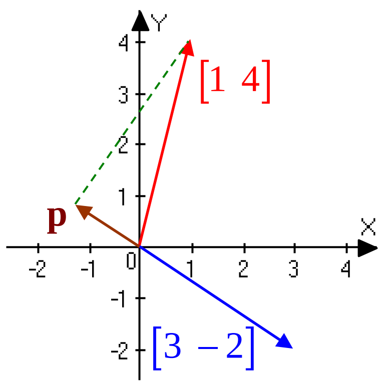
I love getting rid of trig!

Dampen your enthusiasm: the trigonometry is not tossed out, just moved behind the scenes. Moreover, trigonometry is very important and I recommend that you become its friend rather than avoiding it.

Example:

The projection of $[1\ \ 4]$ on $[3\ \ -2]$ is given by:

$$\mathbf{p} = \frac{[1\ \ 4]\bullet[3\ \ -2]}{[3\ \ -2]\bullet[3\ \ -2]}[3\ \ -2]$$
$$= \frac{3-8}{9+4}[3\ \ -2] \approx [-1.15\ \ 0.77]$$



as can be seen from the graph. Notice that since the angle between the two vectors is obtuse, the projection has orientation opposite to $[3\ \ -2]$, though, of course, same direction.

I notice that in all this we have used two sides of a right triangle, but not the third.

Not yet, but here it comes:

Definition

The vector $\mathrm{perp}_{\mathbf{v}}\mathbf{u} = \mathbf{u} - \mathrm{proj}_{\mathbf{v}}\mathbf{u}$ is called the ***component*** of **u** ***orthogonal*** to **v**.

In closing this whole chapter, let me use this third side to provide the first of several *"decomposition theorems"*, that is, ways to split an object into two (or more) pieces that may be used conveniently in certain situations. Decomposition theorems are fairly common in mathematics (think of the prime factorization of a number) and are used in many applications, both theoretical and practical. In particular, this first factorization theorem of the course is a staple in statics: do you remember where you have used it in that course?

Technical fact

Orthogonal decomposition of one vector along another

Given any two vectors **u** and **v** it is possible to express **u** as the ***sum*** of a vector parallel to **v** and a vector perpendicular to **v**, as follows:

$$\mathbf{u} = \mathrm{proj}_{\mathbf{v}}\mathbf{u} + \mathrm{perp}_{\mathbf{v}}\mathbf{u}$$

Example:

We have seen that the projection of $\mathbf{u} = \begin{bmatrix}1 & 4\end{bmatrix}$ on $\mathbf{v} = \begin{bmatrix}3 & -2\end{bmatrix}$ is given by: $\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \begin{bmatrix}-\frac{15}{13} & \frac{10}{13}\end{bmatrix}$. This means that:

$$\mathrm{perp}_{\mathbf{v}}\mathbf{u} = \begin{bmatrix}1 & 4\end{bmatrix} - \begin{bmatrix}-\frac{15}{13} & \frac{10}{13}\end{bmatrix} = \begin{bmatrix}\frac{28}{13} & \frac{42}{13}\end{bmatrix}$$

Notice that by the way we computed it, $\mathbf{u} = \mathrm{proj}_{\mathbf{v}}\mathbf{u} + \mathrm{perp}_{\mathbf{v}}\mathbf{u}$. Moreover, the fact that:

$$\begin{bmatrix}-\frac{15}{13} & \frac{10}{13}\end{bmatrix} \bullet \begin{bmatrix}\frac{28}{13} & \frac{42}{13}\end{bmatrix} = -\frac{15 \times 28}{13^2} + \frac{10 \times 42}{13^2} = 0$$

confirms that the two components of the vector are perpendicular.

Summary

- Orthogonal projections are used in all kinds of applications and their computation through linear algebra methods is very simple.
- The formulae for orthogonal projections are based on the dot product and the information it carries about angles.
- By using orthogonal projections, we can express a vector as the sum of two orthogonal ones, one of which is along a given direction.

Common errors to avoid

- Watch for the details of the formulae, as it is easy to confuse them. But that will not be a good excuse if YOU confuse them in a test!
- Remember that these are *orthogonal* projections. This will prove important in many of the later applications. Also, there are other types of projections, but that is a topic for another course…

Learning questions for Section LA 1-9

Review questions:

1. Explain what an orthogonal projection is and why it is useful.
2. Describe the difference between *projection* and *scalar projection* of a vector on another.
3. Describe what is meant by *the component of a vector orthogonal to another vector*.

Memory questions:

1. Which formula provides the projection of a vector **u** on the vector **v**?
2. Which formula provides the scalar projection of a vector **u** on the vector **v**?
3. Which formula provides the component of a vector **u** that is orthogonal to the vector **v**?
4. Which technical word is equivalent to “perpendicular?”
5. Which technical word is equivalent to “*orthogonal*”?

Computation questions:

1. Determine the projection of the vector $\mathbf{u} = \begin{bmatrix} 3 & -\sqrt{2} & 5 \end{bmatrix}$ on $\mathbf{v} = \begin{bmatrix} 2 & \sqrt{3} & -3 \end{bmatrix}$.
2. Given the vectors $\mathbf{u} = \begin{bmatrix} 1 & 2 & k \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 & 6 & 9 \end{bmatrix}$, determine the values of k for which the projection of **u** on **v** has length 1.

3. Given the vectors $\mathbf{u} = [3 \;\; 2 \;\; -2]$ and $\mathbf{v} = [-1 \;\; 2 \;\; 5]$, construct the projections of **u** and **u+v** on **v** and check algebraically that they are parallel.

4. Find all values of k (if any) for which the projection of the vector $\mathbf{u} = [k \;\; 4 \;\; 2]$ on the vector $\mathbf{v} = [2 \;\; k \;\; 4]$ is:
   a) in the *x*-*z* plane
   b) parallel to $\mathbf{i} + \mathbf{j}$

5. Given the vectors $\mathbf{u} = [1 \;\; 2 \;\; 3]$ and $\mathbf{v} = [2 \;\; -1 \;\; 1]$, write **u** as the sum of a vector parallel to **v** and one orthogonal to it.

6. If we project a vector $\mathbf{v} = [x \;\; y \;\; z]$ on the standard unit vectors **i**, **j** and **k** we should get the usual projections: check that this is so.

Theory questions:

1. When is it true that the projection of a vector **u** on a vector **v** equals **u**?

2. In the formula for the projection of **u** onto **v**, what portion represents the unit vector in the direction of such projection?

Proof questions:

1. Use the formulae that define $\text{proj}_{\mathbf{v}}\mathbf{u}$ and $\text{perp}_{\mathbf{v}}\mathbf{u}$ to show that they are, in fact, perpendicular.

Templated questions:

1. Pick two vectors **u** and **v** and construct the projection of each of them on the other.

2. Pick two vectors **u** and **v** and construct the component of each that is orthogonal to the other.

What questions do you have for your instructor?