Chapter 10: Eigenvalues and diagonalization *Section 1*

Eigenvalues and eigenvectors

What you need to know already:	*What you can learn here:*
➤ Basic properties of linear transformations. ➤ Linear systems and how to solve them. ➤ Determinants and how to compute them.	➤ How to identify certain values and vectors that play a special role for a given matrix.

Given a matrix **A**, the transformation $T_{\mathbf{A}} : \mathbf{x} \to \mathbf{Ax}$ changes each vector in its domain to a possibly different one. In doing so, it may change dimension, or direction, or magnitude or any combination of them.

There is one special question that turns out to be of special interest. Namely, when does the transformation simply stretch the vector **x**? That is, when does it change it to a multiple of itself? Of course, in order to do that the matrix must be a square one, and equally of course, when **A** is square, the **0** vector is not only stretched, but left unchanged by $T_{\mathbf{A}}$.

But which other vectors have this property for a given square matrix **A** and does every square matrix stretch some non-zero vectors, or is this a special property of some matrices only?

We begin, as usual, with some terminology.

Definition

Given an $n \times n$ matrix **A**, a scalar λ and the system $\mathbf{Ax} = \lambda \mathbf{x}$:

- Any value of λ for which the system has ***non-trivial solutions*** is called an ***eigenvalue*** of **A**
- Any non-trivial solution **x** corresponding to an eigenvalue λ of **A** is called an ***eigenvector*** of **A** for λ.

Eigenwhat? What kind of words are these?

They are adapted from the German language, where "*eigen*" identifies a feature that is *owned by* or *specific to* an object or person. So, eigenvalues and eigenvectors can be thought of as specific values and vectors owned by a matrix. We'll see later in what sense this specific association works, but for now, here is an example.

Example: $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

You can check that $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$. Therefore, we can say that the number $\lambda = 2$ is an eigenvalue for this matrix and the vector $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for **A**, corresponding to the eigenvalue $\lambda = 2$.

But in this case the vector was shrunk, not stretched by the matrix!

Yes, but notice that I used the word "*stretch*" only in my informal discussion, not in the formal definition. So, we'll think of shrinking and stretching as the same phenomenon when it comes to eigenvalues. The point is that the vector is changed to one parallel to it, only its magnitude may change.

So, we say that an eigenvector is "specific" to the matrix in the sense that the direction of the vector is somehow special for the matrix, right?

Yes, that is a good way to see it, but you will soon see that there is much more to eigenvalues and eigenvectors than just a special direction.

But now think of another aspect: how did I come up with that eigenvector and eigenvalue for the matrix I used?

Lucky guess? Magic? Advanced linear algebra methods?

None of that: just an application of the properties of determinants. Look at the definition again.

We are looking for solutions of the system $\mathbf{Ax} = \lambda\mathbf{x}$, which can be written as $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$. But the latter is a homogenous system whose matrix of coefficients is $\mathbf{A} - \lambda\mathbf{I}$, so it has non-trivial solutions only when this matrix is not invertible, hence, when its determinant is 0. This gives us a simple way to find eigenvalues.

Strategy for finding the eigenvalues of a square matrix

The eigenvalues of a square matrix **A** are the solutions of the equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

Simple if you know how to solve equations!

True, but by this time I do expect you to be able to solve every equation…

Unfair! Not every equation can be solved: you told us that in calculus!

Touché! But I was about to say that I expect you to be able to solve every equation that I give you and that I have checked to be solvable ☺. And since we are discussing equations, let me give you the name of this special equation that we need to solve in our quest for eigenvalues.

Definition

If **A** is an *n*×*n* matrix, the determinant $|\mathbf{A} - \lambda\mathbf{I}|$ is a polynomial of degree *n* in the variable λ called the ***characteristic polynomial*** of **A**.

Also, the equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$ is a polynomial equation of degree *n* in λ that is called the ***characteristic equation*** of **A**.

Notice that the word "*characteristic*" is another way of rendering the German word "*eigen*" in English. Hopefully, this will help you understand the concept and remember the jargon.

Example: $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

The characteristic polynomial of this matrix is given by:

$$\left| \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 1-\lambda & -1 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & -1 & 1-\lambda \end{vmatrix}$$

By using a Laplace expansion on the second row, this becomes:

$$(2-\lambda)\left[(1-\lambda)^2 - 1\right] = (2-\lambda)\left[\lambda^2 - 2\lambda\right]$$

So, to find the eigenvalues all we have to do is solve this equation by looking at its factors:

$$(2-\lambda)\left[\lambda^2 - 2\lambda\right] = 0 \quad \Rightarrow \quad -\lambda(2-\lambda)^2 = 0 \quad \Rightarrow \quad \lambda = 0, 2$$

Therefore, the only eigenvalues of this matrix are 0 and 2.

I guess I can solve this kind of equations.

I thought so. By the way, in real life applications, the characteristic equation may not always be easy to solve. In that case, numerical methods may be needed to approximate both eigenvalues and eigenvectors.

Before moving on to the issue of how to find eigenvectors, let me point out an interesting fact. I gave the characteristic equation as $|\mathbf{A} - \lambda\mathbf{I}| = 0$. Some books prefer the alternative method of using $|\lambda\mathbf{I} - \mathbf{A}| = 0$.

Isn't that the same thing, since we are using absolute val… Ooops! That's a determinant, not an absolute value!

Right you are, but thinking of absolute values was not such a bad idea, since the two equations are different only in the presence or absence of an extra negative. The alternative form $|\lambda\mathbf{I} - \mathbf{A}| = 0$ has the advantage of giving rise to a polynomial whose highest power of λ always has a coefficient of 1.

Nice and elegant, but when you use it, you are forced to subtract from $\lambda\mathbf{I}$ each and every entry of **A**, which can be long tedious and prone to errors if you forget to do it even just once.

The formulation I use may create a coefficient of -1 in front of the highest power of λ when *n* is odd, but it only requires you to subtract λ from the diagonal entries of **A**. I, and many of my former students like this better. However, try both and use whichever one you prefer.

We'll see. What about eigenvectors? How do we find them?

Once we have an eigenvalue, we go back to the definition. We are looking for solutions of the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, so to find the eigenvectors all we need to do is find such solutions. Just to emphasize it:

Strategy for finding the eigenvectors of a matrix for a given eigenvalue

To find the eigenvectors of **A** corresponding to the eigenvalue λ, we find the solutions of the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ with the usual method of finding the *RREF* of $\mathbf{A} - \lambda\mathbf{I}$ and reading the solutions from it.

I see that we are back to computing an RREF!

They will not easily go away! Let's see how it is done with our good old example matrix.

Example: $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

We found that the eigenvalues of this matrix are 0 and 2. Let's find the eigenvectors for each of them.

For $\lambda = 0$ we have:

$$\mathbf{A} - 0\mathbf{I} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix} \overset{R_3 + R_1}{\Rightarrow} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\overset{R_3 + R_2}{\Rightarrow} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{\frac{1}{2}R_2}{\Rightarrow} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1 + R_2}{\Rightarrow} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the eigenvectors are all multiples of $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$.

For $\lambda = 2$ we have:

$$\mathbf{A} - 2\mathbf{I} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix} \overset{R_3 - R_1}{\Rightarrow} \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y-z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Any linear combination of this type is an eigenvector for this eigenvalue. Notice that if we pick $y = z = 1$ we obtain the eigenvector $\begin{bmatrix} -2 & 1 & 1 \end{bmatrix}$ that I used in the original example.

And let us now have a look at a different matrix, which also has an additional convenient feature.

Example: $\begin{bmatrix} 3 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

This matrix is special because of the large number of 0's it contains. This is sometimes called a *sparse* matrix and these matrices are often found in applications.

When it comes to finding its eigenvalues, it is nice because we can break the search down into smaller pieces by noticing that any eigenvalue must be an eigenvalue for the top left or the bottom right 2×2 minors.

So, we use the salami technique and look at the two minors separately:

$$\begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$$

Therefore, the only eigenvalue is 2, but it appears twice. We'll look at this phenomenon in the next section. For now, let us find its eigenvectors, but for that we need to consider the whole matrix!

$$\begin{bmatrix} 3-2 & 1 & 0 & 0 \\ -1 & 1-2 & 0 & 0 \\ 0 & 0 & 1-2 & -4 \\ 0 & 0 & 1 & 1-2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$RREF \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, the eigenvectors are all multiples of $\begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix}$.

For the other minor, we have:

$$\begin{vmatrix} 1-\lambda & -4 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5$$

This polynomial has no roots, therefore there are no eigenvalues from this minor! Notice that I never said that a matrix *must* have eigenvalues, since that is not true, as this example shows.

Nice, but what is all this good for?

Oh, the utility question again! Can't something be worth studying just because it is there and it is interesting?

Not when you are still looking forward to getting a job!

Point well taken. As I said earlier, there are many uses of eigenvalues and eigenvectors. For instance, the well-known search program called Google, uses them to figure out which pages are most likely to meet your needs when you tell it which topic you are looking for.

Cool! How does that work?

Cool it: unfortunately, this is a more advanced use of the concept and you need to learn a few things beyond this course to get there. Other similar examples that require more advanced math knowledge include pattern recognition, analysis of object stability, risk analysis in economics, study of vibrations, statistics, control theory and many more!

But here is a small application that I can present easily.

Technical fact

The effect of multiplying a matrix **A** by one of its eigenvectors is to simply ***stretch*** it by an amount equal to the eigenvector.

Well, that just follows from the definition, right?

Yes. If **x** is an eigenvector, $\mathbf{Ax} = \lambda \mathbf{x}$, so x gets stretched by a factor of λ.

And finally, here are two interesting facts that are very useful, but so easy to prove that I leave such proof as part of your *Learning questions* work.

Technical fact

If λ is an eigenvalue for an invertible matrix **A**, then $\frac{1}{\lambda}$ is an ***eigenvalue for*** $\mathbf{A}^{-1}$ and with the ***same eigenvectors***.

If λ is an eigenvalue for an invertible matrix **A**, then λ^k is an ***eigenvalue for*** $\mathbf{A}^k$ and with the ***same eigenvectors***.

Summary

- The eigenvalues of a matrix **A** are the solutions of the characteristic equation $|\mathbf{A} - \lambda \mathbf{I}| = \mathbf{0}$.
- The eigenvectors of a matrix **A** for the eigenvalue λ are those vectors that, when multiplied by **A** are changed to λ**A.**
- Eigenvalues are eigenvectors have many practical applications, most of which require mathematical and other knowledge beyond the scope of this course.

Common errors to avoid

- None that are specific to eigenvalues, but remember that to find eigenvalues and eigenvectors you need to solve polynomial equations and linear systems: make sure you are familiar with doing both!

Learning questions for Section LA 10-1

Review questions:

1. Explain what an eigenvalue and its eigenvectors are.
2. Describe how to compute eigenvalues.
3. Describe how to compute the eigenvectors for a given eigenvalue.

Memory questions:

1. What is the geometric interpretation of an eigenvector?
2. How are the eigenvalues of $\mathbf{A}$ and $\mathbf{A}^{-1}$ related?
3. What equation is called the *characteristic equation* of a matrix *A*?

Computation questions:

For each of the matrices provided in questions 1-13, determine the characteristic equation, the eigenvalues and the corresponding eigenvectors.

1. $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 6 \\ 0 & 8 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

5. $\begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ 1 & -3 & -2 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 0 & 1 \end{bmatrix}$

9. $\begin{bmatrix} -1 & 0 & 0 \\ -6 & 5 & -2 \\ -3 & 3 & -2 \end{bmatrix}$

10. $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & 2 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 2 & 4 & 6 \\ -3 & 0 & 1 \\ 0 & 5 & 8 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 6 & 3 \\ 0 & -5 & -3 \\ 0 & 2 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 5 & 4 & 3 & 2 \\ -1 & 1 & -1 & 1 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

14. Given the matrix $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & -3 & -2 \end{bmatrix}$:

  a) Determine the value of k for which the matrix is not invertible.

  b) Determine the eigenvalues of this matrix, if any, for such value of k.

15. Determine the eigenvalues of the matrix $\mathbf{A}^5$, where $\mathbf{A} = \begin{bmatrix} 3 & 0 & 3 \\ 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$

Theory questions:

1. What is the characteristic equation of I_2 ?
2. What is the main connection between determinants and eigenvalues?
3. Does a 3×3 matrix always have an eigenvalue?
4. If **A** has no eigenvalue, what can we say about its determinant?
5. If a matrix **B** has 0 as an eigenvalue, can it be invertible?
6. Which eigenvalue is common to all non-invertible matrices?
7. What portion of the characteristic polynomial is the determinant of its matrix?
8. What can one possibly mean by the *eigenvectors of a linear transformation*?
9. If **A** is a square matrix for which $\mathbf{A}^7 = \mathbf{A}$, what are its possible eigenvalues?
10. If **A** is a square matrix for which $\mathbf{A}^8 = \mathbf{A}$, what are its possible eigenvalues?

Proof questions:

1. Prove that if λ is an eigenvalue for an invertible matrix **A**, then $\frac{1}{\lambda}$ is an eigenvalue for $\mathbf{A}^{-1}$ and with the same eigenvectors.
2. Prove that if λ is an eigenvalue for a square matrix **A**, then λ^k is an eigenvalue for $\mathbf{A}^k$ and with the same eigenvectors.
3. Show that the characteristic equation of a matrix of the form $\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ is $\lambda^4 = a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$. Explain why this implies that any monic polynomial of degree 4 is characteristic for some 4×4 matrix. (You may want to search what "*monic*" means!)
4. Prove that the eigenvalues of an upper or lower diagonal matrix are its diagonal entries and determine what the corresponding eigenvectors are.
5. Prove that if **A** is a square matrix, λ is one of its eigenvalues, and **u** and **v** are two eigenvectors of λ, then any linear combination of **u** and **v** is also an eigenvector for **A** and λ.
6. Prove that the set of eigenvectors of a matrix for one of its eigenvalues is a subspace.
7. Show that if $p(\lambda)$ is the characteristic polynomial of a matrix **A**, than $|\mathbf{A}| = p(0)$.
8. Prove that the constant of the characteristic polynomial of a matrix equals its determinant.
9. An $n \times n$ matrix $\mathbf{A}$ is said to be *idempotent* if $\mathbf{A}^2 = \mathbf{A}$. Prove that if $\mathbf{A}$ is an idempotent matrix, then its only possible eigenvalues are 0 and 1.
10. An $n \times n$ matrix $\mathbf{A}$ is said to be *nilpotent* if $\mathbf{A}^n = \mathbf{0}_n$. Prove that if $\mathbf{A}$ is a nilpotent matrix, then its only possible eigenvalue is 0.

11. Identify all $n \times n$ matrices for which all n-dimensional vectors are eigenvectors.

Application questions:

12. The transformation determined by the matrix $C = \begin{bmatrix} 2 & 1/3 \\ 3 & 2 \end{bmatrix}$ changes every point on the line $3x - y = 0$ to another point on the same line.

a) Show that this is in fact true.
b) Determine the other line through the origin that has the same property for this matrix.
c) Use accurate technical language to describe the relationship between the matrix and the vectors on the line $3x - y = 0$.

Templated questions:

1. Construct a small square matrix, find its eigenvalues and, for each of them find the corresponding eigenvectors.

2. For invertible matrix **A** for which you know the eigenvalues, compute the eigenvalues of $\mathbf{A}^{-1}$ and $\mathbf{A}^{2}$.

What questions do you have for your instructor?