Chapter 10: Eigenvalues and diagonalization — **Section 2**

Eigenspaces

What you need to know already:	What you can learn here:
➤ What the eigenvalues and eigenvectors of a matrix are and how to find them.	➤ The structure of properties of the sets of eigenvectors for each eigenvalue.

You are having fun with this eigenstuff, eh?

Yes, and so should you! Seriously, though, the title of the section should suggest that it is about jargon.

Jargon? In linear algebra? What a surprise!

We have seen before that the eigenvectors of a matrix $\mathbf{A}$ that correspond to a particular eigenvalue λ are the solutions of the homogeneous system $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$. But this means that such vectors form the null space of the matrix $\mathbf{A}-\lambda\mathbf{I}$. That deserves highlighting in a definition!

Definition

The ***set of eigenvectors*** for a matrix $\mathbf{A}$ corresponding to an eigenvalue λ consists of the null space of the matrix $\mathbf{A}-\lambda\mathbf{I}$ and hence forms a subspace that is called the ***eigenspace*** of λ.

We have also seen before that to identify the eigenvectors of a given eigenvalue we just solve the system $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0}$. This same method also provides a basis for the eigenspace, so that we can identify such eigenspace completely.

OK! On to the next section…

Not so fast! There is an important consideration related to the dimension of this eigenspace, one that leads not only to more jargon, but to interesting information about the matrix itself, as we shall see in later sections.

Notice that each eigenvalue is a root of the characteristic polynomial $|\mathbf{A}-\lambda\mathbf{I}|=0$. Let me remind you of a concept in basic algebra that you have probably not used much for a while.

Definition

If c is a ***solution*** of a polynomial equation of the form $p(x)=0$, then $p(x)$ can be written in the form:

$$p(x)=(x-c)^n q(x)$$

for some integer number n.

The largest value of n for which this is possible is called the ***algebraic multiplicity*** of c.

Example: $x^3 - 2x^2 + x = 0$

This equation can be written as $(x^2 - 2x + 1)x = 0$, which tells us that 0 is a solution of the equation. Since we cannot factor out any other power of x, 0 is a solution of algebraic multiplicity 1.

However, we can continue factoring the left side into $(x-1)^2 x = 0$. This tells us that 1 is also a solution of this equation, with multiplicity 2, since the factor $(x-1)$ is raised to the second power.

Since eigenvalues are solutions of the characteristic equation of a matrix, the following definition is natural.

Definition

If $\mathbf{A}$ is a square matrix and λ is one of its eigenvalues, the ***algebraic multiplicity*** of λ is its algebraic multiplicity as a solution of the characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

But we have just seen that the eigenvectors of an eigenvalue form a Euclidean subspace, and as such it has a dimension. Let's give it this number a name as well.

Definition

If $\mathbf{A}$ is a matrix and λ is one of its eigenvalues, the ***geometric multiplicity*** of λ is the ***dimension of its eigenspace***.

Example: $\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 0 & 1 \end{bmatrix}$

The characteristic equation of this matrix is given by:

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 3 & -1-\lambda & 3 \\ 2 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (-1-\lambda)\left[(1-\lambda)^2 - 4\right] = 0$$

$$\Rightarrow (1+\lambda)(\lambda^2 - 2\lambda - 3) = 0$$

$$\Rightarrow (1+\lambda)(\lambda-3)(\lambda+1) = 0 \Rightarrow (\lambda+1)^2(\lambda-3) = 0$$

Therefore, this matrix has two eigenvalues: $\lambda = -1$, with algebraic multiplicity 2, and $\lambda = 3$, with algebraic multiplicity 1.

To find their geometric multiplicities, we look for their eigenspaces.

For $\lambda = -1$ we get:

$$\begin{bmatrix} 1-\lambda & 0 & 2 \\ 3 & -1-\lambda & 3 \\ 2 & 0 & 1-\lambda \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 0 & 3 \\ 2 & 0 & 2 \end{bmatrix} \begin{matrix} R_2 - 1.5R_1 \\ \Rightarrow \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The two free variables tell us that its eigenspace has also dimension 2 and therefore the geometric multiplicity of this eigenvalue is also 2. You may want to get a basis for this eigenspace, for the sake of practice.

For $\lambda = 3$ we get:

$$\begin{bmatrix} 1-\lambda & 0 & 2 \\ 3 & -1-\lambda & 3 \\ 2 & 0 & 1-\lambda \end{bmatrix} = \begin{bmatrix} -2 & 0 & 2 \\ 3 & -4 & 3 \\ 2 & 0 & -2 \end{bmatrix} \begin{matrix} R_2 + 1.5R_1 \\ \Rightarrow \\ R_3 + R_1 \end{matrix} \begin{bmatrix} 2 & 0 & 2 \\ 0 & -4 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

This time we only have one free variable, so that the eigenspace has also dimension 1 and therefore the geometric multiplicity of this eigenvalue is also 1. Again, you may want to get a basis for this eigenspace, for the sake of practice.

Hey: I see a connection: it seems that algebraic and geometric multiplicity of an eigenvalue are the same number! Is that so?

One would think that, based on this example, right? But let me give you another example.

Example: $\begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$

This matrix has only one eigenvalue, $\lambda = 3$, with algebraic multiplicity 3. You may want to check this yourself: never trust your teachers completely, especially when it gives you a chance to practice!

Let's find its geometric multiplicity:

$$\begin{bmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ -1 & 0 & 2-\lambda \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ \Rightarrow \\ R_3 + R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As you can see, the geometric multiplicity is now 2, NOT 3!

OK, I stand corrected! But is there ever a relation between the two multiplicities?

Technical fact

The ***geometric*** multiplicity of an eigenvalue is ***never larger*** than the ***algebraic*** multiplicity.

Cool! And how do we prove that?

With arguments based on more advanced technical facts that we have not seen yet and will not help you in this course. So, here you do have to trust me. Or, you may search for the proof in the literature and figure out what those advanced facts are. ☺

I gladly trust you, for now…

And to thank you for such trust, here is an interesting fact about eigenspaces that is fairly "*easy*" to prove.

Technical fact

If $\mathbf{A}$ is a square matrix, and $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is a set of non-zero eigenvectors for it, each corresponding to a ***different*** eigenvalue, then the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ are linearly ***independent***.

Proof

We start small, and prove this fact for two eigenvectors corresponding to two different eigenvalues. So, let's say that $\lambda_1 \neq \lambda_2$, $\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$ and $\mathbf{A}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2$. Can the eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ be dependent? If true, that would mean that they are multiples of each other, say $\mathbf{v}_1 = c\mathbf{v}_2$. But that would imply that, at the same time:

$$\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1 = \lambda_1 c\mathbf{v}_2 \quad \text{and} \quad \mathbf{A}\mathbf{v}_1 = \mathbf{A}c\mathbf{v}_2 = c\mathbf{A}\mathbf{v}_2 = c\lambda_2 \mathbf{v}_2$$

But two multiples of the same vector can be equal only if the scalars are equal, so that we need:

$$c\lambda_1 = c\lambda_2 \quad \Rightarrow \quad \lambda_1 = \lambda_2$$

But we assumed that the two eigenvalues are not equal, hence the eigenvectors cannot be dependent.

OK, what about larger sets of eigenvectors? In that case, we can use the method of induction, that is, we can show that if the statement is true for *k* eigenvectors, it must be true for a larger set. This is not difficult, but long and tedious, so I will spare you the experience. As usual, you can look it up in the literature, but you need to look hard: few authors have the patience to go through all the steps!

Summary

- The set of eigenvectors for an eigenvalue of a matrix forms a subspace, called eigenspace, whose dimension is called the geometric multiplicity of the eigenvalue.
- The algebraic multiplicity of the eigenvalue is its multiplicity as a solution of the characteristic equation.
- The geometric multiplicity is never larger than the algebraic multiplicity.

Common errors to avoid

- It is easy to get confused between geometric and algebraic multiplicity and to forget which one is supposed to be larger. Therefore, keep looking these ideas up until you are comfortable with them.

Learning questions for Section LA 10-2

Review questions:

1. Describe the difference between algebraic and geometric multiplicity of an eigenvalue.

Memory questions:

1. What is the algebraic multiplicity of an eigenvalue?
2. What is the geometric multiplicity of an eigenvalue?
3. Which multiplicity of an eigenvalue is never smaller than the other?

Computation questions:

For each of matrices provided in questions 1-24, determine the algebraic and geometric multiplicity of each eigenvalue and a basis for its eigenspace.

1. $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 6 \\ 0 & 8 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & -3 \\ 1 & 2 & -2 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ 1 & -3 & -2 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 0 & 1 \end{bmatrix}$

11. $\begin{bmatrix} -1 & 0 & 0 \\ -6 & 5 & -2 \\ -3 & 3 & -2 \end{bmatrix}$

12. $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & 2 & 0 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 4 & 6 \\ -3 & 0 & 1 \\ 0 & 5 & 8 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 6 & 3 \\ 0 & -5 & -3 \\ 0 & 2 & 2 \end{bmatrix}$

15. $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 4 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

17. $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & 2 & 0 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$

20. $\begin{bmatrix} 0 & 2 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

21. $\begin{bmatrix} -1 & 0 & 3 & 0 \\ 3 & 5 & 1 & 5 \\ -2 & 0 & 6 & 0 \\ 1 & 0 & -3 & 0 \end{bmatrix}$

22. $\begin{bmatrix} 0 & 3 & 0 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

23. $\begin{bmatrix} 5 & 4 & 3 & 2 \\ -1 & 1 & -1 & 1 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

24. $\begin{bmatrix} 7 & -19 & 25 & -16 & 4 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Theory questions:

1. If **A** is an $n \times n$ matrix and λ is an eigenvalue with geometric multiplicity of 2, what is the rank of $\mathbf{A}-\lambda\mathbf{I}$?

2. Is a linear combination of eigenvectors an eigenvector?

3. What is another name for the nullity of the matrix $B-\lambda I$ when looked at from the perspective of eigenvalues?

4. If you find that the only eigenvector for a certain eigenvalue is the zero vector, what can you conclude?

5. Is it possible for an eigenspace to have dimension 0?

6. Is the sum of two eigenvectors an eigenvector?

7. If a 3×3 matrix **A** has three distinct eigenvalues and I pick one eigenvector for each of them, what is the linear relationship among them?

Proof questions:

1. Prove that a matrix **A** has two distinct eigenvalues and we pick one eigenvector for each of them, any non-zero linear combination of them will not be an eigenvector.

2. In this section I claimed that an eigenspace is a subspace because it is the null space of a certain matrix. Prove that it is a subspace by checking the requirements of the definition of subspace directly.

3. Prove that the eigenvalues of any matrix of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ are $\lambda = a, c$ and determine the corresponding eigenspaces for all possible values of a, b and c.

Templated questions:

1. Construct a square matrix not larger than 4×4, compute its eigenvalues and, for each of them, determine its algebraic and geometric multiplicity and a basis for its eigenspace.

What questions do you have for your instructor?