Chapter 10: Eigenvalues and diagonalization — **Section 3**

Diagonal matrices

What you need to know already:	*What you can learn here:*
➤ Basic definition, properties and operations of matrix.	➤ The key properties of diagonal matrices that make them "nice".

By now we have done many things with matrices: developed algebra operations, used them to solve systems, to determine basis and dimensions, eigenvectors, etc. We have also run into many of their difficulties: lack of commutativity in the product, difficulty of computation for inverses and eigenspaces, etc.

In this section we shall look more closely at a type of matrices that are nice, in the sense that they have very nice properties that reduce or eliminate some of the above difficulties. Of course I am referring to diagonal matrices, and just in case you forgot, here is their definition again.

Definition

A square matrix **A** is said to be a ***diagonal*** matrix if its only non-zero entries are those along the main diagonal.

That is, a matrix **A** is ***diagonal*** if $a_{ij} = 0$ whenever $i \neq j$.

I remember them. Aren't they kind of boring?

It depends on what you mean by boring. I see them as being nice, and I will provide several arguments supporting my claim.

Ok, I am curious about them. But I am also curious about why in the middle of a chapter that started on eigenstuff, you now turn to diagonal matrices. Is there a connection, or should this chapter better be called "odds and ends"?

Oh, there is a connection! In fact it is a strong and very surprising one. But in order to lead you to it I have to prepare the ground by discussing the properties of diagonal matrices first. And when the connection will be ready to appear, I guarantee that it will be a big surprise indeed. But don't peek ahead, or you'll spoil it ☺.

OK, so what are these great properties?

Here is the first.

Technical fact

The ***product*** of two $n \times n$ diagonal matrices is done ***entry by entry*** and generates an $n \times n$ ***diagonal*** matrix.

Proof

We notice that the dot product of the *i*-th row of a diagonal matrix with the *j*-th row of another diagonal matrix is given by:

$$[0 \cdots a_i \cdots 0] \bullet [0 \cdots b_j \cdots 0]$$

This is different from 0 only if $i = j$, otherwise all terms of the dot product are 0. Therefore, their product is:

$$\begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} a_1b_1 & 0 & \cdots & 0 \\ 0 & a_2b_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & a_nb_n \end{bmatrix}$$

So, aren't powers of diagonal matrices also diagonal?

Certainly, since powers are product. But there is more.

Technical fact

The ***product*** of diagonal matrices is ***commutative***.

Proof

That is an immediate consequence of the fact that for diagonal matrices the product is done entry-by-entry, by using the usual product.

And here is another simple and nice property.

Technical fact

The ***determinant*** of a diagonal matrix is given by the ***product of its diagonal entries***.

Proof

If no diagonal entry of the matrix is 0, then the matrix is in *REF* form and the conclusion follows.

If one of the diagonal entries is 0, then its row is the zero vector and the conclusion follows again.

Technical fact

The ***eigenvalues*** of a diagonal matrix are its ***diagonal entries***.

The ***algebraic multiplicity*** of each eigenvalue equals the ***number of times*** such value appears on the diagonal.

No need for a proof: I can see that!

I thought so. But just for fun, I will give you a proof of the next fact.

By the way, is this the surprising connection?

Oh no, this is just a little and trivial intermediate step. The surprise will come in a later section.

Oh, patience again, eh?

Technical fact

The ***eigenvectors*** of an eigenvalue of a diagonal matrix can have any scalar in the positions occupied in the matrix by that eigenvalue, but must have 0's elsewhere.

Proof

If **A** is diagonal and λ is one of its eigenvalues, the entries of the matrix $\mathbf{A}-\lambda\mathbf{I}$ are 0 off the diagonal, as well as on all diagonal positions occupied by the eigenvalue. The other positions will have a non-zero value, so that their corresponding variable in the homogeneous system will need to have a value of 0, while the variable in the eigenvalues positions are free.

And since for each occurrence of the eigenvalue we have one different scalar, which we can see as a free variable, it follows that:

Technical fact

For any eigenvalue of a diagonal matrix, ***the algebraic multiplicity is equal to the geometric multiplicity***.

It sounds convincing, but very abstract and possibly confusing. Can we see an example?

Example: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

The following matrix is diagonal and its eigenvalues are 1, 2 and 3.

To find the eigenvectors of $\lambda = 1$, we look at the matrix $\mathbf{A}-\lambda\mathbf{I}$:

$$\mathbf{A}-\mathbf{I} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Notice that what have accomplished is to set all the diagonal entries that equaled the eigenvalue to now equal 0. (And if we had any zeros on the diagonal before, they would now be non-zero entries)

By seeing this as representing a homogeneous system, we conclude that the last four variables must equal 0, while the first one is free (no leading coefficient). Therefore, the eigenvectors are of the form $\begin{bmatrix} x & 0 & 0 & 0 & 0 \end{bmatrix}$, as described in the statement.

For the eigenvalue 2, the matrix $\mathbf{A}-\lambda\mathbf{I}$ is:

$$\mathbf{A}-2\mathbf{I} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Again, we notice that we have a free variable, this time the fourth one, where the eigenvalue was, while all others must equal 0. Therefore, the eigenvectors are $\begin{bmatrix} 0 & 0 & 0 & x & 0 \end{bmatrix}$.

Finally, for the eigenvalue 3, we have:

$$\mathbf{A}-3\mathbf{I}=\begin{bmatrix}-2&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&-1&0\\0&0&0&0&0\end{bmatrix}$$

This time the first and fourth variables must equal 0, while the others are all free. Therefore, its eigenvectors are of the form $\begin{bmatrix}0 & x & y & 0 & z\end{bmatrix}$.

I see! It did look messy, but it is very simple!

That is the whole point: diagonal matrices are nice!

I wish all matrices were like that: it sure would make things easier!

So does everyone else, and that is why in the next section we'll try to develop a way to change matrices to a diagonal form.

"Try not: Do or do not!"

Alas, this is a situation where even Yoda would agree to compromise. The method we shall develop will make many matrices diagonal, but not all of them. Some matrices simply prefer to remain complicated.

The point of this section was to bring this important type of matrices to your attention. Practicing on them is extremely simple, so the *Learning questions* are very few.

Yeah! Long live diagonal matrices!

Summary

- Diagonal matrices are very nice! What else is there to say? Well, review the details of the section, this is a summary! ☺

Common errors to avoid

- Just because diagonal matrices have certain nice properties, do not think every matrix has them, nor that no other matrix does! That's why they are special, but not unique.

Learning questions for Section LA 10-3

Review questions:

1. Identify the nice properties that hold for diagonal matrices.

Memory questions:

1. Is the product of diagonal matrices commutative?
2. How are diagonal matrices multiplied together?
3. What are the eigenvalues of a diagonal matrix?
4. What number provides the algebraic multiplicity of an eigenvalue of a diagonal matrix?

Theory questions:

1. What is the geometric multiplicity of an eigenvalue of a diagonal matrix?
2. Mention two nice properties of diagonal matrices.

Templated questions:

1. Construct a small diagonal matrix (no larger than 6×6) and compute the eigenspace for each of its eigenvalues. You may want to use a random number generator to decide on the diagonal entries.

What questions do you have for your instructor?