Chapter 10: Eigenvalues and diagonalization *Section 4*

Similarity

What you need to know already:	*What you can learn here:*
➢ Matrix product and inverses. ➢ Determinants. ➢ Linear transformations. ➢ Definition of eigenvalues and eigenspaces.	➢ A very useful relationship among square matrices that preserved many important properties of the matrices involved.

In the section on diagonal matrices I claimed that there is a way to make certain square matrices diagonal, in some sense, although not all of them. In this section I will describe the tool we use for that purpose.

But we know a way already: change it into an RREF by using Gauss-Jordan elimination!

Good suggestion, but that is not what we need. Remember that diagonal matrices are nice because they have nice properties. Gauss-Jordan does not preserve any of those properties. As good and useful as it is, it is not what we are looking for here. And moreover, it would leave us with diagonal matrices whose diagonal entries can only be 0 or 1. We need to look in another direction, so as to get something useful and interesting.

The key is in the following procedure.

Definition

If $\mathbf{P}$ is an invertible $n\times n$ matrix, the transformation defined by:

$$d_{\mathbf{P}}(\mathbf{A}) = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

is called a ***similarity***.

Example: $\mathbf{P} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Since this matrix is invertible and $\mathbf{P}^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$, the transformation:

$$d_{\mathbf{P}}(\mathbf{A}) = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix} \mathbf{A} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

is a similarity. For instance:

$$d_{\mathbf{P}}\left(\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -15 & -24 \\ 12 & 19 \end{bmatrix}$$

Why a "similarity?" What is it similar to?

The transformation is not similar, but it changes a matrix to a similar one.

Still, in what way?

Through the matrix properties it preserves. But let's start from some properties of the transformation itself.

Technical fact

A similarity is a ***linear transformation***, when viewing matrices as vectors.

Proof

We need to show that it preserves linear combinations. By definition:

$$d_{\mathbf{P}}(h\mathbf{A} + k\mathbf{B}) = \mathbf{P}^{-1}(h\mathbf{A} + k\mathbf{B})\mathbf{P}$$

By using the distributivity property of matrix multiplication, this becomes:

$$= \mathbf{P}^{-1}h\mathbf{A}\mathbf{P} + \mathbf{P}^{-1}k\mathbf{B}\mathbf{P} = h\mathbf{P}^{-1}\mathbf{A}\mathbf{P} + k\mathbf{P}^{-1}\mathbf{B}\mathbf{P}$$
$$= hd_{\mathbf{P}}(\mathbf{A}) + kd_{\mathbf{P}}(\mathbf{B})$$

Therefore, the linear structure is preserved.

If it is a linear transformation, isn't it a matrix transformation? So, shouldn't a single matrix on one side be enough to define it?

That is true, but what kind of matrix? When we view an $n \times n$ matrix as a vector, its dimension is n^2, so that the matrix you are referring to would be a gigantic $n^2 \times n^2$ that would make us want to quit and go for dinner. Moreover, it would not help us at all in discovering the other properties, since we want to maintain the matrix structure and its properties.

So, we stick with our definition and keep looking.

Technical fact

A similarity ***preserves matrix products***, in the sense that:

$$d_{\mathbf{P}}(\mathbf{AB}) = d_{\mathbf{P}}(\mathbf{A})d_{\mathbf{P}}(\mathbf{B})$$

Proof

Let us check by direct verification, that is, by using the definition:

$$d_{\mathbf{P}}(\mathbf{AB}) = \mathbf{P}^{-1}(\mathbf{AB})\mathbf{P} = \mathbf{P}^{-1}\mathbf{AIBP}$$

But we can write $\mathbf{I}$ as the product of $\mathbf{P}$ and its inverse:

$$= \mathbf{P}^{-1}\mathbf{APP}^{-1}\mathbf{BP} = (\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{BP}) = d_{\mathbf{P}}(\mathbf{A})d_{\mathbf{P}}(\mathbf{B})$$

As claimed!

I can see that this similarity preserves a lot: it looks like salt!

Yes, but there is more…

Technical fact

A similarity ***preserves determinants***, in the sense that:

$$\left|d_{\mathbf{P}}(\mathbf{A})\right| = |\mathbf{A}|$$

Proof

This follows from the fact that the determinant preserves products:

$$\left|d_{\mathbf{P}}(\mathbf{A})\right| = \left|\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\right| = \left|\mathbf{P}^{-1}\right||\mathbf{A}||\mathbf{P}| = \frac{1}{|\mathbf{P}|}|\mathbf{A}||\mathbf{P}| = |\mathbf{A}|$$

Technical fact

A similarity ***preserves characteristic equations***, in the sense that:

$$\left|d_{\mathbf{P}}(\mathbf{A}) - \lambda\mathbf{I}\right| = 0 \quad \Leftrightarrow \quad |\mathbf{A} - \lambda\mathbf{I}| = 0$$

Proof

This follows again from the fact that the determinant preserves products:

$$\left|d_{\mathbf{P}}(\mathbf{A}) - \lambda\mathbf{I}\right| = \left|\mathbf{P}^{-1}\mathbf{A}\mathbf{P} - \lambda\mathbf{I}\right|$$
$$= \left|\mathbf{P}^{-1}\mathbf{A}\mathbf{P} - \lambda\mathbf{P}^{-1}\mathbf{I}\mathbf{P}\right| = \left|\mathbf{P}^{-1}(\mathbf{A} - \lambda\mathbf{I})\mathbf{P}\right|$$
$$= \left|\mathbf{P}^{-1}\right||\mathbf{A} - \lambda\mathbf{I}||\mathbf{P}| = |\mathbf{A} - \lambda\mathbf{I}|$$

But if it preserves characteristic equations, it also preserves eigenvalues, which are its solutions!

Yes, of course! So, just for the record:

Technical fact

A similarity ***preserves eigenvalues***, in the sense that $d_{\mathbf{P}}(\mathbf{A})$ and $\mathbf{A}$ have the same eigenvalues.

We are now ready to shift our focus from the transformation to the matrices.

Definition

Two $n \times n$ matrices **A** and **B** are said to be ***similar*** if there is an invertible $n \times n$ matrix **P** such that:

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{B} \quad \Leftrightarrow \quad \mathbf{P}\mathbf{B}\mathbf{P}^{-1} = \mathbf{A}$$

And based on what we have said so far, the following fact is immediate.

Technical fact

If two matrices are ***similar***, then they have the ***same determinant***, the ***same characteristic equation***, and the same ***eigenvalues***.

What about eigenvectors? Are they the same too?

No, but almost.

Technical fact

If λ is an eigenvalue of the matrix **A**, **v** is one of its ***eigenvectors*** and $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a matrix similar to **A**, then $\mathbf{P}^{-1}\mathbf{v}$ is an ***eigenvector*** for **B** with eigenvalue λ.

Proof

The assumptions tell us that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. Therefore:

$$\mathbf{B}\left(\mathbf{P}^{-1}\mathbf{v}\right) = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\left(\mathbf{P}^{-1}\mathbf{v}\right) = \mathbf{P}^{-1}\mathbf{A}\mathbf{v} = \mathbf{P}^{-1}\lambda\mathbf{v} = \lambda\left(\mathbf{P}^{-1}\mathbf{v}\right)$$

This confirms the claim.

What happened to diagonal matrices? I thought we were trying to get them.

You are right, but in this section we only developed the tool we need to make a matrix diagonal. In the next one we'll see how to use this tool.

In fact, this section, like the previous one, has the main purpose of introducing some terminology, so the *Learning questions* are again fairly basic.

Summary

- A similarity transformation is a linear transformation obtained through an invertible matrix and it changes a matrix to a similar one that has the same determinant and eigenvalues, with eigenvectors corresponding in a simple way.

Common errors to avoid

- Just don't get lost in the many features that are preserved by a similarity: not everything is preserved by them!
- A similarity is just a tool used, later on, to transform some matrices to diagonal versions. It is useful, but not a panacea.

Learning questions for Section LA 10-4

Review questions:

1. Describe what a similarity is.

2. Identify the properties that are preserved by a similarity transformation.

Memory questions:

1. What is the general formula of a similarity transformation?

2. Is a similarity transformation linear?

Computation questions:

1. Determine the matrix that is similar to $\mathbf{A} = \begin{bmatrix} 2 & 4 & 1 \\ 1 & 2 & 4 \\ 4 & 1 & 2 \end{bmatrix}$ through $\mathbf{P} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and verify that it has the same determinant and eigenvalues as **A**.

2. Determine the matrix that is similar to $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ through $\mathbf{P} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ and verify that it has the same determinant and eigenvalues as **A**.

Theory questions:

1. If two square matrices are similar, how are their determinants?

2. What feature of the matrices $\mathbf{I}_n$ and $2\mathbf{I}_n$ tells us, by visual inspection, that they are not similar?

3. If you are given **A** and **P**, what is the easiest way to compute the determinant of $\mathbf{P}^{-1}\mathbf{AP}$ by hand?

4. If a matrix **A** is similar to **B**, does it always follow that **B** is similar to **A**?

5. Similarity is a linear transformation, so can it be defined by a single matrix, instead of using $\mathbf{P}$ and $\mathbf{P}^{-1}$?

Proof questions:

1. Provide a sufficient reason why the matrices $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ are not similar.

2. Identify all matrices (if any) that are similar to the identity matrix.

3. If **Q** is an orthogonal matrix, under what conditions is a matrix similar to it also orthogonal?

Templated questions:

1. Construct an invertible 3×3 matrix **P**, then the similarity transformation it defines and apply it to another 3×3 matrix.

What questions do you have for your instructor?