Chapter 10: Eigenvalues and diagonalization **Section 5**

Diagonalization

What you need to know already:	*What you can learn here:*
➤ Properties of diagonal matrices. ➤ The concept of similarity. ➤ How to identify eigenvalues and eigenvectors of a matrix.	➤ How to construct a similarity between a given matrix and a diagonal matrix. ➤ How to determine if such similarity even exists.

We are now ready to attempt to make a matrix almost diagonal, but first let me clarify what we mean by that.

Definition

A matrix $\mathbf{A}$ is ***diagonalizable*** if it is ***similar*** to a diagonal matrix $\mathbf{D}$, that is, if there is an invertible matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}=\mathbf{D}$.

Example: $\mathbf{A}=\begin{bmatrix}3&2\\0&1\end{bmatrix}$

Notice that, since $\mathbf{P}=\begin{bmatrix}-1&1\\1&0\end{bmatrix}$ is invertible, $\mathbf{P}^{-1}=\begin{bmatrix}0&1\\1&1\end{bmatrix}$ and:

$$\mathbf{P}^{-1}\begin{bmatrix}3&2\\0&1\end{bmatrix}\mathbf{P}=\begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}3&2\\0&1\end{bmatrix}\begin{bmatrix}-1&1\\1&0\end{bmatrix}=\begin{bmatrix}1&0\\0&3\end{bmatrix}$$

it follows that the matrix $\mathbf{A}$ is diagonalizable.

By using all the properties of diagonal matrices and of similarity, it is fairly easy to prove the following facts, which are left for your enjoyment in the *Learning questions*.

Technical fact

If a matrix $\mathbf{A}$ is ***diagonalizable*** and similar to the diagonal matrix $\mathbf{D}$, then:

- The ***determinant*** of $\mathbf{A}$ is the product of the diagonal entries of $\mathbf{D}$.

- The ***eigenvalues*** of **A** are the diagonal entries of **D**, each with geometric multiplicity given by the number of times that value appears in the diagonal.

What about the other properties of diagonal matrices, such as commutativity of product, multiplication entry-by-entry and so on?

I am afraid we cannot rely on them in a simple way. ☹ After all, we are talking about *similar* matrices, not identical twins! Sometimes in life you have to be content with little gains and aim for the bigger ones at a later stage.

Don't get philosophical on me! But, back to what we are trying to do, how do we figure out if a matrix is diagonalizable?

I am glad you asked. Obviously if we start from a diagonal matrix **D** and an invertible matrix **P** and construct $\mathbf{P}^{-1}\mathbf{AP}=\mathbf{D}$, we end up with a diagonalizable matrix. But what you are asking is the more interesting question of checking whether a given matrix **A** is diagonalizable, without knowing which diagonal matrix it is similar to or which invertible matrix to use for the similarity.

To move in that direction, we start from a simple, but important observation.

Technical fact

If **A** is a ***diagonalizable*** matrix and λ is one of its eigenvalues, then the algebraic and geometric ***multiplicity*** of λ must be ***equal***.

Proof

Given $\mathbf{P}^{-1}\mathbf{AP}=\mathbf{D}$, we can use three facts we know:

- For any eigenvalue of **D**, algebraic and geometric multiplicities are the same.
- The eigenvalues of **D** are the same as those of **A**.
- If **v** is an eigenvector of **D** for λ, $\mathbf{P}^{-1}\mathbf{v}$ is an eigenvector of **A**

With this is mind, if λ is a (common) eigenvalue of multiplicity *n*, there are *n* independent vectors $\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}$ that are eigenvectors for **D**. But then $\{\mathbf{P}^{-1}\mathbf{v}_1,\cdots,\mathbf{P}^{-1}\mathbf{v}_n\}$ are eigenvectors for **A** and, since $\mathbf{P}^{-1}$ is invertible, they are independent. Therefore, the geometric multiplicity is at least *n*. But then again, it cannot be greater than *n*, so it must equal *n*.

And here is another important fact whose proof follows immediately from what we have seen so far.

Technical fact

If **A** is a diagonalizable matrix, with $\mathbf{P}^{-1}\mathbf{AP}=\mathbf{D}$, the diagonal entries of **D** are the eigenvalues of **A**, each repeated as many times as its multiplicity:

$$\mathbf{P}^{-1}\mathbf{AP}=\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}$$

But we also know that eigenvectors coming from different eigenvalues are independent and we know that a diagonalizable matrix has, for each eigenvalue, as many independent eigenvectors as the multiplicity. That means that an $n\times n$ diagonalizable matrix has *n* independent eigenvectors. But the opposite is true!

Technical fact

If the $n \times n$ matrix $\mathbf{A}$ has n independent eigenvectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, then it is diagonalizable, with the similarity obtained by setting:

$$\mathbf{P} = [\mathbf{v}_1 \cdots \mathbf{v}_n], \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

where λ_i is the eigenvalue of the corresponding $\mathbf{v}_i$.

Proof

Since the given vectors are independent, the matrix $\mathbf{P}$ obtained by using them as columns is invertible. So, what we need to show is that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$. Let us check:

$$\mathbf{PD} = [\mathbf{v}_1 \cdots \mathbf{v}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [\lambda_1 \mathbf{v}_1 \cdots \lambda_n \mathbf{v}_n] =$$

$$= [\mathbf{Av}_1 \cdots \mathbf{Av}_n] = \mathbf{A}[\mathbf{v}_1 \cdots \mathbf{v}_n] = \mathbf{AP}$$

But this means that:

$$\mathbf{PD} = \mathbf{AP} \quad \Rightarrow \quad \mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$$

as claimed.

Time out: You seem to be doing a lot of playing with of lot of symbols here! I do not really understand what is going on!

I can see why you may have missed the main event in the forest of details I just gave you. So, let me restate what we figured out theoretically as a strategy. After you are comfortable with the strategy, please go back through the previous theoretical facts and see how they all fit.

Strategy for finding a diagonalization of a square matrix

To determine if and how an $n \times n$ matrix $\mathbf{A}$ is diagonalizable:

1. ***Find*** the eigenvalues of $\mathbf{A}$ and their eigenspaces.
2. If one of the eigenvalues has geometric ***multiplicity less*** than its algebraic multiplicity, then $\mathbf{A}$ is ***NOT*** diagonalizable.
3. Otherwise, ***select*** n independent eigenvectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, by picking a basis for each eigenspace.
4. Finally, ***know that*** $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, where:

$$\mathbf{P} = [\mathbf{v}_1 \cdots \mathbf{v}_n], \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

where λ_i is the eigenvalue of $\mathbf{v}_i$.

Would you like an example?

Please!!!! And a cup of coffee, to stay awake!

Example: $\mathbf{A}=\begin{bmatrix} 8 & 4 \\ -1.5 & 3 \end{bmatrix}$

Let us find the eigenvalues of this matrix. The characteristic polynomial:

$$(8-\lambda)(3-\lambda)+6=\lambda^2-11\lambda+30=(\lambda-5)(\lambda-6)$$

tells us that they are $\lambda=5,6$. Now we look for their eigenvectors.

$$\lambda=5 \quad\Rightarrow\quad \mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix} 3 & 4 \\ -1.5 & -2 \end{bmatrix} \quad\Rightarrow\quad \mathbf{x}_5=\begin{bmatrix} -4t \\ 3t \end{bmatrix}=t\begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$\lambda=6 \quad\Rightarrow\quad \mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix} 2 & 4 \\ -1.5 & -3 \end{bmatrix} \quad\Rightarrow\quad \mathbf{x}_6=\begin{bmatrix} -2t \\ t \end{bmatrix}=t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

We are now ready to put it all together. All we need to do is take care that the eigenvectors are placed in the same positions as their eigenvalues:

$$\mathbf{D}=\begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix},\ \mathbf{P}=\begin{bmatrix} -4 & -2 \\ 3 & 1 \end{bmatrix}$$

I leave to you and your calculator the task of check that $\mathbf{P}^{-1}\mathbf{AP}=\mathbf{D}$.

Is that why in the definition of similarity the inverse of $\mathbf{P}$ *comes first, because the one that consists of eigenvectors comes later in the formula?*

Exactly! When you first see the formula for similarity it may look strange why the inverse comes first, but that is the reason: the eigenvectors are the columns of the later matrix in the formula and the ones from which we start to construct the diagonalization.

And to reward you for this splendid observation, here is the another example.

Example: $\mathbf{A}=\begin{bmatrix} 1 & 6 & 3 \\ 0 & -5 & -3 \\ 0 & 2 & 2 \end{bmatrix}$

The eigenvalues of this matrix are given by:

$$\begin{vmatrix} 1-\lambda & 6 & 3 \\ 0 & -5-\lambda & -3 \\ 0 & 2 & 2-\lambda \end{vmatrix}=$$

$$=(1-\lambda)\left[(-5-\lambda)(2-\lambda)+6\right]=(1-\lambda)\left(\lambda^2+3\lambda-4\right)$$

$$=(1-\lambda)(\lambda+4)(\lambda-1)=0 \quad\Rightarrow\quad (1-\lambda)^2(\lambda+4)=0$$

Therefore, they are $\lambda=1$, with algebraic multiplicity 2, and $\lambda=-4$, with multiplicity 1. Now we look for their eigenvectors.

$$\lambda=1 \quad\Rightarrow\quad \begin{bmatrix} 0 & 6 & 3 \\ 0 & -6 & -3 \\ 0 & 2 & 1 \end{bmatrix} \underset{R_3-R_1/3}{\overset{R_2+R_1}{\Rightarrow}} \begin{bmatrix} 0 & 6 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow\quad \mathbf{x}_1=\begin{bmatrix} x \\ -z/2 \\ z \end{bmatrix}=x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}+z\begin{bmatrix} 0 \\ -.5 \\ 1 \end{bmatrix}$$

$$\lambda=-4 \quad\Rightarrow\quad \begin{bmatrix} 5 & 6 & 3 \\ 0 & -1 & -3 \\ 0 & 2 & 6 \end{bmatrix} \overset{R_3+2R_2}{\Rightarrow} \begin{bmatrix} 5 & 6 & 3 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{-R_2}{\Rightarrow} \begin{bmatrix} 5 & 6 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1-6R_2}{\Rightarrow} \begin{bmatrix} 5 & 0 & -15 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{x}_{-4} = \begin{bmatrix} 3z \\ -3z \\ z \end{bmatrix} = z\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$$

We are now ready to put it all together. All we need to do is take care that the eigenvectors are placed in the same positions as their eigenvalues. One way to do that is the following:

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}, \ \mathbf{P} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -.5 & -3 \\ 0 & 1 & 1 \end{bmatrix}$$

I leave to you and your calculator the task of check that $\mathbf{P}^{-1} = \begin{bmatrix} 1 & 1.2 & .6 \\ 0 & .4 & 1.2 \\ 0 & -.4 & -.2 \end{bmatrix}$

and that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$.

But why did you place those eigenvalues and their eigenvectors in that order?

No particular reason, other that this was the order in which I analyzed them. This is an important fact to keep in mind.

Knot on your finger

A diagonalizable matrix can be ***diagonalized in infinitely many*** different ways, depending on:

- how we ***arrange*** the eigenvalues on the diagonal, and
- how we ***pick*** the eigenvectors for each eigenvalue.

So, for instance…

Example: $\mathbf{A} = \begin{bmatrix} 1 & 6 & 3 \\ 0 & -5 & -3 \\ 0 & 2 & 2 \end{bmatrix}$

Given what we know about this matrix, we can set up another diagonalization by changing the order of the eigenvalues:

$$\mathbf{D} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{P} = \begin{bmatrix} 3 & 0 & 1 \\ -3 & -.5 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

or by picking different eigenvectors for them:

$$\mathbf{D} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{P} = \begin{bmatrix} -3 & 0 & -1 \\ 3 & 1 & 0 \\ -1 & -2 & 0 \end{bmatrix}$$

It's now your turn to try this strategy by applying it to some simple matrices.

Summary

- A matrix is diagonalizable if it is similar to a diagonal matrix, in which case the eigenvalues and determinant can be obtained from the simpler diagonal matrix.
- An $n \times n$ matrix is diagonalizable if it has n independent eigenvectors, which occurs is the geometric multiplicities of all eigenvalues add up to n.

Common errors to avoid

- Don't get scared by the many links presented in this section and by their somewhat abstract substance. Try enough examples to see how the strategy actually works in practice and those links should become clearer.

Learning questions for Section LA 10-5

Review questions:

1. Explain what *diagonalizing a matrix* means.
2. Describe how to determine whether a matrix is diagonalizable.
3. Describe how to diagonalize a matrix when it can be done.

Memory questions:

1. Which matrices are diagonalizable?
2. Which matrices are not diagonalizable?
3. Is there only one way to diagonalize a matrix?

Computation questions:

Construct a diagonalization for each of the matrices presented in questions 1-8. That means, given the matrix **A**, construct the matrices **P** and **D** such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$

1. $\begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

3. $\begin{bmatrix} -1 & 0 & 0 \\ -6 & 5 & -2 \\ -3 & 3 & -2 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 6 & 3 \\ 0 & -5 & -3 \\ 0 & 2 & 2 \end{bmatrix}$

9. Construct a 2×2 matrix **A** by knowing that it has two eigenvalues, $\lambda_1 = 3$ with eigenspace $k\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\lambda_2 = -3$ with eigenspace $h\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

10. Determine the values of k for which the matrix $\mathbf{A} = \begin{bmatrix} 1 & 4 & -2 \\ 0 & -5 & -3 \\ 0 & k & 2 \end{bmatrix}$ is diagonalizable

11. Determine a matrix **A** whose characteristic polynomial is $8 + 4\lambda - 2\lambda^2 - \lambda^3$ and for which one of the eigenvectors of its positive eigenvalue is [1 1 1].
NOTE: There are several possible correct answers.

Theory questions:

1. If a matrix is diagonalizable, what can we say about the relation between algebraic and geometric multiplicities of its eigenvalues?

2. If a matrix is diagonalizable, what is the sum of the geometric multiplicities of all its eigenvalues?

3. If a matrix is diagonalizable, can its determinant be written in terms of eigenvalues?

4. Is every diagonalizable matrix invertible?

5. If a matrix is diagonalizable, do its rows form a basis for its row space?

6. According to the definition, is a diagonal matrix diagonalizable?

7. A 3×3 matrix **A** has eigenvalues 1, -1 and 2, with corresponding eigenvectors $\begin{bmatrix}1 & 0 & 0\end{bmatrix}$, $\begin{bmatrix}1 & 3 & 2\end{bmatrix}$ and $\begin{bmatrix}1 & 1 & 1\end{bmatrix}$. Construct an invertible matrix P and a diagonal matrix D such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$.

Proof questions:

1. Prove that if **A** is diagonalizable, so is its transpose and determine eigenvalues and eigenvectors of the transpose.

Templated questions:

1. Construct a small matrix, determine whether it is diagonalizable and, if so, construct one such diagonalization,

What questions do you have for your instructor?