Inner products

What you need to know already:	*What you can learn here:*
➤ The concept of vector space. ➤ Euclidean dot product and its consequences.	➤ How the dot product and its consequences may be extended to general vector spaces.

When we extended the definition of vector space to sets of objects other than usual Euclidean vectors, we chose the key features that made the structure "*tick*" algebraically, thus selecting the axioms that allowed us to carry through the idea of linear combinations. From that flowed many properties such as bases and dimensions, linear independence etc.

But that still leaves out a key property of vectors that gave us so many useful tools and facts, namely the dot product. In this section, you'll see how to extend the dot product to vectors that are not Euclidean.

Hold on: I can see why the dot product was useful with Euclidean vectors, given that it allowed us to define matrix product, matrix transformations, eigenvalues and so on. But what good would it do to extend it to more general vector spaces?

The point is to see if we can extend certain geometric concepts to non-geometric settings. Remember that, in addition to the uses you just mentioned, the dot product also allowed us to develop easy algebraic ways to compute geometric quantities such as:

- the length of a vector: $\|\mathbf{v}\| = \sqrt{\mathbf{v} \bullet \mathbf{v}}$
- the angle between two vectors: $\theta = \cos^{-1}\left(\dfrac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)$
- the perpendicular projection of a vector on another: $proj_{\mathbf{v}}\mathbf{u} = \dfrac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$.

So, if we can generalize the dot product, we can also get access to these concepts in a more general way. Of course, we'll also have to see if these generalizations are actually meaningful and fruitful, but mathematicians would have abandoned this idea if it had proved useless!

In fact. the trick to achieve such a generalization is once again to identify the main *properties* that make the dot product useful, rather than look at its definition, and then focus on them. It turns out that we only need four properties to preserve the good features of the dot product, and they are as follows.

Definition

Given a vector space $\mathfrak{V}$, an ***inner product*** is an ***operation*** that:

- associates to each ***pair of vectors*** **u** and **v** in $\mathfrak{V}$ a ***real number***, denoted by $\langle \mathbf{u}, \mathbf{v} \rangle$, and
- has the following properties, called ***inner product axioms***:

1. ***Commutativity***:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$$

2. ***Distributivity***:

$$\langle \mathbf{u} + \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle$$

3. ***Associativity***:

$$k \langle \mathbf{u}, \mathbf{v} \rangle = \langle k\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, k\mathbf{v} \rangle$$

4. ***Positive definition***: $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, with the equality $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ occurring if and only if $\mathbf{u} = \mathbf{0}$.

Of course, the familiar dot product satisfies these axioms: can you hear a *Proof question* calling? But here are two examples of inner products, one for a Euclidean vector space, but different from the usual dot product, and one for a different vector space.

Example: *Weighted dot product in* $\mathbb{R}^2$

The usual dot product in $\mathbb{R}^2$ considers both components as equally important, in that they are used in the same way in the formula.

But what if we are using an application in which the two components have different impact, for some reason? In that case, we can consider a different inner product defined by:

$$\langle [u_1 \;\; u_2], [v_1 \;\; v_2] \rangle = au_1v_1 + bu_2v_2$$

where a and b are two positive numbers. Is this a legitimate inner product? Let's check the axioms:

Commutativity follows from the fact that usual multiplication is commutative.

For *distributivity* we have:

$$\langle [u_1 \;\; u_2] + [w_1 \;\; w_2], [v_1 \;\; v_2] \rangle = \langle [u_1 + w_1 \;\; u_2 + w_2], [v_1 \;\; v_2] \rangle$$

$$a(u_1 + w_1)v_1 + b(u_2 + w_2)v_2 = au_1v_1 + aw_1v_1 + bu_2v_2 + bw_2v_2$$

$$= au_1v_1 + bu_2v_2 + aw_1v_1 + bw_2v_2 = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle$$

Associativity can be proved in the same way.

As for *positive definition*, we have that:

$$\langle \mathbf{u}, \mathbf{u} \rangle = au_1^2 + bu_2^2$$

Since we chose a and b to be positive this is always positive and can only be 0 for the zero vector.

This shows that this is a proper inner product, usually referred to as the ***weighted dot product***.

Isn't this what we get by first stretching the vectors' components and then applying the dot product?

Yes! Can you figure out by how much we must stretch each component?

Example: *Integral product on* $\mathcal{C}_I$

All continuous functions are integrable and we have seen that their set forms a vector space with the usual operations. We can define an inner product on this space by setting:

$$\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)\,dx$$

But is it really an inner product? That is, are all four axioms satisfied? Let's check them by remembering the basic properties of definite integrals.

Commutativity: $\int_0^1 f(x)g(x)\,dx = \int_0^1 g(x)f(x)\,dx$ by the commutativity of usual multiplication

Distributivity: All we need is usual properties of integrals:

$$\int_0^1 \left(f_1(x)+f_2(x)\right)g(x)\,dx = \int_0^1 \left(f_1(x)g(x)+f_2(x)g(x)\right)dx$$

$$= \int_0^1 f_1(x)g(x)\,dx + \int_0^1 f_2(x)g(x)\,dx \quad \text{as required}$$

Associativity is true since we can move a coefficient anywhere in or out of the integral.

Positive definition is also true since the integral must be positive if the integrand is such:

$$\langle f(x), f(x)\rangle = \int_0^1 f^2(x)\,dx \ge 0$$

Moreover, the equality is true if and only if the integrand is the zero function. Therefore, we are dealing with a true inner product.

There are many other examples, some of which are waiting for you as Learning questions. But now that we have our tool, let us reap its fruits.

Definition

In a vector space $\mathcal{V}$ endowed with an inner product $\langle \mathbf{u}, \mathbf{v}\rangle$, we can ***define*** the following generalization of the corresponding familiar concepts.

- The ***norm*** of a vector **u** is:

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u}\rangle}$$

- The ***angle*** between two vectors **u** and **v** is:

$$\theta = \cos^{-1}\left(\frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)$$

- Two vectors **u** and **v** are ***orthogonal*** if:

$$\langle \mathbf{u}, \mathbf{v}\rangle = 0$$

- The ***orthogonal projection*** of **u** on **v** is:

$$proj_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{v}\|^2}\mathbf{v}$$

Ok, so have made sense of these concepts in an abstract context. Now what?

First of all, do you notice any weak points in these definitions?

They all look fine! No, wait! In order for the angle to be defined, we must make sure that the argument of the inverse cosine is between -1 and 1!

That is right: if not, some vectors may not form a so-called angle at all! But instead of showing you that this is indeed the case, I will claim a more general and more famous fact. Its proof is so much fun that I have left it for you in the *Learning questions*.

Technical fact
The Cauchy-Schwarz inequality

In a vector space $\mathcal{V}$ endowed with an inner product $\langle \mathbf{u}, \mathbf{v}\rangle$, it is always true that $|\langle \mathbf{u}, \mathbf{v}\rangle| \le \|\mathbf{u}\|\|\mathbf{v}\|$.

And since this is true, it follows that the fraction within the inverse cosine is never greater than 1 and so the angle is well defined!

Bingo! And notice that this important property follows just from the four axioms!

Example: $\langle [u_1 \ \ u_2],[v_1 \ \ v_2] \rangle = 3u_1v_1 + u_2v_2$

Let us consider this weighted dot product in $\mathbb{R}^2$. We can notice that $[0 \ \ 1]$ is still a unit vector, since:

$$\left\| [0 \ \ 1] \right\| = \sqrt{3\times 0^2 + 1^2} = 1$$

However, the vector $[1 \ \ 0]$ is not, since:

$$\left\| [1 \ \ 0] \right\| = \sqrt{3\times 1^2 + 0^2} = \sqrt{3}$$

And are these two vectors still orthogonal, since they are so in the usual dot product? Let's see:

$$\langle [1 \ \ 0],[0 \ \ 1] \rangle = 3\times 1\times 0 + 0\times 1 = 0$$

So, yes, they are! But we also know that with the usual dot product the vectors $[1 \ \ 0]$ and $[1 \ \ 1]$ form an angle of $\pi/4$. What is that angle in this weighted product? We use the defining formula:

$$\theta = \cos^{-1}\left(\frac{\langle [1 \ \ 0],[1 \ \ 1] \rangle}{\left\|[1 \ \ 0]\right\| \left\|[1 \ \ 1]\right\|} \right) = \cos^{-1}\left(\frac{3\times 1\times 1 + 0\times 1}{\sqrt{3}\sqrt{3+1}} \right)$$

$$= \cos^{-1}\left(\frac{3}{2\sqrt{3}} \right) = \cos^{-1}\left(\frac{\sqrt{3}}{2} \right) = \frac{\pi}{6}$$

The fact that this inner product overemphasizes the first component makes these two vectors closer in angle!

If that is the case, what is the projection of $[1 \ \ 1]$ on $[1 \ \ 0]$? With the usual dot product it is $[1 \ \ 0]$ itself, but does the weighting change things? Make a guess before we check algebraically:

$$proj_{[1 \ 0]}[1 \ \ 1] = \frac{\langle [1 \ \ 1],[1 \ \ 0] \rangle}{\left\|[1 \ \ 0]\right\|^2}[1 \ \ 0] = \frac{3}{3}[1 \ \ 0] = [1 \ \ 0]$$

Still the same! Can you figure out why, perhaps by looking back at what we found earlier in the example?

That is weird indeed! But what's the point?

For now, just to make you realize that the usual dot product and the usual notion of angle are not entirely "*natural*", in the sense that there are alternatives that work equally well mathematically. Let's state that formally.

Technical fact

A vector space may be endowed with several different inner products.

All properties that are true for an inner product are true for all possible ones.

Therefore, it may be useful, if not necessary, to clearly identify which inner product one is using on a particular inner space before interpreting the consequences of such properties.

In terms of applications, I am afraid I will have to delay that again to after you have learned more about both inner products and general vector spaces and their uses.

So, for now, use them as a fun way to enlarge, expand and – dare I say it? – generalize the concepts of norm and angle that you thought you knew and understood!

And isn't it nice to find a good old friend while exploring this strange and weird situation?

I wish! But where is it?

Well, it is the always faithful companion of mathematical explorations.

Technical fact
The Pythagorean theorem for general vector spaces

In a vector space $\mathfrak{V}$ endowed with an inner product, if **u** and **v** are two orthogonal vectors, then:

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Let me guess: you are leaving the proof to me!

Of course! Why would I deny you such a great pleasure? But here is how it works in our last, simple example.

Example: $\langle [u_1 \;\; u_2], [v_1 \;\; v_2] \rangle = 3u_1v_1 + u_2v_2$

We have seen that with this inner product, $[0 \;\; 1]$ is a unit vector and $[1 \;\; 0]$ has length $\sqrt{3}$. So, the sum of the squares of their lengths is 4.

Moreover, we checked that they are still orthogonal. So, will the Pythagorean theorem work? Let's see:

$$\|[0 \;\; 1]+[1 \;\; 0]\|^2 = \|[1 \;\; 1]\|^2 = \langle [1 \;\; 1], [1 \;\; 1] \rangle$$
$$= 3\times 1\times 1 + 1\times 1\times 1 = 4$$

Surprise! It does! Well, hopefully it was not a surprise, but it is comforting to know that it does.

In fact, think about it: haven't we really shown that the Pythagorean theorem works because the usual dot product is an inner product?

What do you mean?

Well, I bet you thought that the Pythagorean theorem works based on a geometric property, but now we have seen that it works because of an algebraic property! Or are they related? And if so, how? And which other properties that we know about the dot product are valid for any inner product?

Can you see the cornucopia of deep and interesting questions this raises?

So may questions, so little time! Oh well, I hope you will take more courses during which you can explore all of them: happy exploring!

Summary

- An operation that associates to any two vectors of a vector space a scalar, in such a way to satisfy four key axioms, is called an inner product.
- When an inner product is chosen for a vector space, it is possible to define the concepts of magnitude, angle, projection, etc. It is also possible to confirm several properties, including the Pythagorean theorem.

Common errors to avoid

- Just because we did not have time in this course to explore all the uses of inner products, both theoretical and practical, it does not mean that there are few or that there are useless. So, instead of dismissing this concept, decide to learn more about it!

Learning questions for Section LA 11-10

Review questions:

1. Describe the underlying concept for the idea of an inner product.

2. List the 4 axioms of an inner product.

Memory questions:

1. Which three key concepts related to the dot product can be obtained through any inner product?

2. Which key geometric theorem is true in any inner product?

Computation questions:

1. In the vector space consisting of 2×2 matrices, consider the inner product defined by:

$$\left\langle \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix}, \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} \right\rangle = \mathbf{a}_1 \bullet \mathbf{b}_1 + \mathbf{a}_2 \bullet \mathbf{b}_2$$

where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2$ are the rows of the matrices.

a) Show that the required axioms of inner products work.

b) Compute the cosine of the angle between the matrices $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ with respect to this inner product.

c) With this inner product, would an orthogonal matrix be also a matrix on norm 1?

2. In the vector space of integrable functions, the integral $\int_0^1 f(x)g(x)\,dx$ defines an inner product. By using it, determine the norm of the function $y = x$ and whether it is orthogonal to $y = x^2 + 1$.

3. Verify that the Pythagorean theorem applies to the vectors $f(x) = x + 1$ and $g(x) = 9x - 5$ of $\mathcal{P}_1$ with the inner product given by $\langle f, g \rangle = \int_0^1 fg\,dx$.

4. In the vector space $\mathcal{P}_1$ of linear functions, the integral $\int_0^1 f(x)g(x)\,dx$ defines an inner product. Determine which vectors are in the "unit circle" for this vector space, that is, which functions have a distance of 1 from $y = 0$.

5. In the vector space of integrable functions, the integral $\int_0^1 f(x)g(x)\,dx$ defines an inner product. Determine the angle formed in the space by the vectors $y = \sin x$ and $y = \cos x$.

Theory questions:

1. What is the purpose of defining inner products?
2. If a vector space is endowed with an inner product, is it always possible to define the magnitude of a vector?
3. If a vector space is endowed with an inner product, is it always possible to define an angle between two vectors?
4. If a vector space is endowed with an inner product, is it always possible to implement the Gram-Schmidt process?
5. Can the usual dot product in $\mathbb{R}^2$ be used to define an inner product in $\mathbb{C}$?
6. Does every vector space have an inner product?
7. Can a vector space have more than one possible inner product?
8. Why is it that the product $\langle [u_1, u_2], [v_1, v_2] \rangle = \sqrt{u_1 v_1 + u_2 v_2}$ cannot be an inner product?
9. Is it true that for an inner product $c\langle \mathbf{u}, \mathbf{v} \rangle = \langle c\mathbf{u}, c\mathbf{v} \rangle$?
10. In a vector space, is it always true that $(\mathbf{u}+\mathbf{v})^2 = \mathbf{u}^2 + \mathbf{uv} + \mathbf{vu} + \mathbf{v^2}$?
11. In a vector space endowed with an inner product, is it always true that $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2$?

Proof questions:

1. Prove that the Pythagorean theorem works in any vector space with an inner product.
2. Prove that if two non-zero vectors in a vector space are orthogonal with respect to some inner product, they are linearly independent.
3. Prove that in any vector space with an inner product, opposite vectors form an angle of π.
4. Prove the Cauchy-Schwarz inequality $|\langle \mathbf{u}, \mathbf{v} \rangle| \le \|\mathbf{u}\|\|\mathbf{v}\|$ by completing the following steps:
   a) Use the inner product axioms to show that the inequality is true when one of the two vectors is the zero vector.
   b) Assuming that $\mathbf{u} \ne \mathbf{0}$, use the inner product axioms to show that for any real number t it is true that:
   $$\langle t\mathbf{u}+\mathbf{v}, t\mathbf{u}+\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle t^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle t + \langle \mathbf{v}, \mathbf{v} \rangle \ge 0$$

c) Explain why part b) implies that the discriminant of the quadratic expression used there is not positive.
d) Compute such discriminant, set it to be non-positive and from that arrive at the Cauchy-Schwarz inequality.

5. Prove that if $\mathcal{V}$ is a vector space with an inner product and $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is an orthogonal basis for one of its subspaces, then $\|\mathbf{u}+\mathbf{v}+\mathbf{w}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$.

6. Prove that if $\mathcal{V}$ is a vector space with an inner product and **u** and **v** are two of its vectors, then the law of cosines holds.

7. The set of diagonal matrices of dimension $n \times n$ forms a vector space. Determine which of these two operations (if any) is an inner product for this space and justify your conclusions.
   a) $\langle \mathbf{A}, \mathbf{B} \rangle = Trace(\mathbf{AB}) = \sum_{i=1}^{n} (\mathbf{AB})_{ii}$
   b) $\langle \mathbf{A}, \mathbf{B} \rangle = |\mathbf{AB}|$

8. Determine whether among complex numbers, multiplication by the conjugate, $\langle z_1, z_2 \rangle = z_1 \overline{z}_2$, provides an inner product.

9. Prove that the weighted dot product is an inner product for any vector with positive components and for any $\mathbb{R}^n$.

10. Prove that if **A** is an invertible $n \times n$ matrix, the formula $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{Au} \cdot \mathbf{Av}$ defines an inner product in $\mathbb{R}^n$ and explain why it is important that the matrix be invertible.

Templated questions:

1. Look up in a textbook or online for a vector space with an inner product and verify that all axioms involved are satisfied.

What questions do you have for your instructor?