Basic definitions and operations

What you need to know already:	What you can learn here:
➤ How to perform all basic operations with real numbers. ➤ The basic properties of algebraic operations on real numbers.	➤ What imaginary and complex numbers are. ➤ How complex numbers are combined through basic algebraic operations.

So, what's a complex number?

Oh, just a really complicated one, you know! Just kidding. Let's start from the beginning.

One of the most notable features of square roots, which anyone who has learn about them knows, is that you cannot compute the square root of a negative number. But, what if we could? We certainly cannot do that by using real numbers, since every real number, when squared provides a positive number. So, we need to invent new, non-real numbers. You might think that the way to do it is to invent a new number corresponding to $\sqrt{-1}$, but it turns out that this approach leads to some technical glitches, so we take a slightly different approach, one so similar that right now you may not see the difference.

> **Definition**
>
> The ***imaginary unit i*** is assumed to be a number with the property that $i^2 = -1$.

You are right: I don't see the difference.

As I said, it is a tiny technical difference and to appreciate it you need to first learn a little more about imaginary and complex numbers.

But this definition just gives us an imaginary unit. To go further is easy.

> **Definition**
>
> An ***imaginary number*** is any entity of the form ki, where k is any real number.

So, an imaginary number is just a multiple of i, right?

Absolutely. So, $2i, \frac{2}{3}i, \pi i, \sqrt{3}i$ and so on, are all imaginary numbers.

At this point we could define how to add, multiply and do operations on these numbers, but that is best done after defining complex numbers.

Definition

A ***complex number*** is an expression of the form $a+bi$, where a and b are real numbers.

It is customary to use letters from the *bottom of the alphabet*, such as ***z*** and ***w***, to identify a complex number, although this is not required, as long as the context is clear.

Example:

The following are examples of complex numbers:

$$2+3i, \quad \sqrt{5}-\pi i, \quad \frac{1}{3}-6i$$

Obviously, you can make up infinitely many more!

As you can see, these complex numbers are not really complicated, only complex in the sense that they combined real and imaginary numbers. In fact…

Definition

Given a complex number $z=a+bi$, the real number ***a*** is called ***the real part of z*** and the real number ***b*** is called the ***imaginary part of z*** and are usually denoted by $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ respectively.

Example:

Well, just for the record, and to recap, the following are real numbers:

$$2, -3, \frac{1}{4}, 5\pi, \sqrt{6}, e^7$$

The following are imaginary numbers:

$$2i, \ -3i, \ \frac{1}{4}i, \ 5\pi i, \ \sqrt{6}i, \ e^7 i$$

The following are complex numbers:

$$2-3i, \quad \frac{1}{4}+5\pi i, \quad \sqrt{6}+e^7 i$$

And, for the last two:

$$\operatorname{Re}\left(\frac{1}{4}+5\pi i\right)=\frac{1}{4} \quad ; \quad \operatorname{Im}\left(\sqrt{6}+e^7 i\right)=e^7$$

And to show you that complex numbers are not that complicated to work with, here is how to add and multiply them.

Definition

Complex numbers are added and subtracted as if they were linear functions in the variable *i*. So, for addition:

$$(a_1+b_1 i)+(a_2+b_2 i)=(a_1+a_2)+(b_1+b_2)i$$

For multiplication, by using the definition of *i*:

$$(a_1+b_1 i)(a_2+b_2 i)=a_1a_2+a_1b_2 i+a_2b_1 i+b_1b_2 i^2$$
$$=(a_1a_2-b_1b_2)+(a_1b_2+a_2b_1)i$$

Example: $2-3i, \quad 5+6i$

Here is a basic example of operations with complex numbers:

$$(2-3i)+(5+6i)=7+3i$$

$$(2-3i)(5+6i)=10+12i-15i-18i^2=28-3i$$

Do subtraction, division and powers follow in the same way?

Almost. For subtraction and integer powers, we operate as usual.

Example: $2-3i, \quad 5+6i$

Here is a basic example of subtraction and squaring:

$$(2-3i)-(5+6i)=-3-9i$$

$$(2-3i)^2=4-12i+9i^2=-5-12i$$

However, for division and roots things are little more delicate. Here are a couple of facts to illustrate the difficulties.

Technical fact

For any positive integer n:

$$i^{2n}=\begin{cases}1 & \text{if } n \text{ even}\\ -1 & \text{if } n \text{ odd}\end{cases} \qquad \left(i^{2n}\right)^2=1$$

Technical fact

For any positive integer n:

$$i^{2n+1}=\begin{cases}i & \text{if } n \text{ even}\\ -i & \text{if } n \text{ odd}\end{cases}$$

$$\left(i^{2n+1}\right)^2=-1 \ ; \ \left(i^{2n}\right)^2=1$$

This means that $\sqrt{-1}$ could be provided by either i or $-i$, thus partially explaining the way the definition of i is given.

Proof

We just need to check the claims:

$$i^{2n+1}=i^{2n}i=(-1)^n i=\begin{cases}i & \text{if } n \text{ even}\\ -i & \text{if } n \text{ odd}\end{cases}$$

$$i^{2n}=(-1)^n=\begin{cases}1 & \text{if } n \text{ even}\\ -1 & \text{if } n \text{ odd}\end{cases}$$

$$\left(i^{2n+1}\right)^2=(\pm i)^2=(\pm 1)^2(i)^2=-1$$

$$\left(i^{2n}\right)^2=(\pm 1)^2=1$$

Notice that, unlike the case for real numbers, where we have a natural preference for the positive ones, here we are making up the new number i, so that we need to define it in terms of what it does, rather than where it comes from, since the latter may lead to ambiguity.

And of course, things get even more complicated when we look at more general roots. We'll stay away from that until we are ready to give a better definition of root.

As for division, since we defined the product by mimicking what we do with polynomials, the problem is that we do not have a general process that starts from two polynomials and divides them into a third polynomial. Fortunately, there is another feature of complex numbers that gives us a solution to the problem.

Definition

The ***conjugate*** of a complex number $z = a + ib$ is the complex number $\overline{z} = a - ib$.

Example:

Here are the conjugates of some good old friends of ours:

$$z = 2 - 3i \quad \Rightarrow \quad \overline{z} = 2 + 3i$$
$$z = \frac{1}{4} + 5\pi i \quad \Rightarrow \quad \overline{z} = \frac{1}{4} - 5\pi i$$
$$z = \sqrt{6} + e^7 i \quad \Rightarrow \quad \overline{z} = \sqrt{6} - e^7 i$$

Notice that this is the same idea behind the conjugate of expressions involving radicals or trigonometric functions. One more important and familiar concept.

Definition

The ***modulus*** of a complex number is defined as:

$$|z| = \sqrt{a^2 + b^2}$$

That looks like the length of a vector!

Yes, and it is, once we make the proper connections in the next section. But also, notice the following interesting and fairly obvious fact.

Technical fact

The product of a complex number z with its conjugate provides the square of the modulus of z:

$$z\,\overline{z} = (a + ib)(a - ib) = a^2 + b^2$$

Example:

Here are the *moduli* (plural of modulus!) of our good old friends:

$$|2 - 3i| = \sqrt{4 + 9} = \sqrt{13}$$
$$\left|\frac{1}{4} + 5\pi i\right| = \sqrt{\frac{1}{16} + 25\pi^2} \quad ; \quad \left|\sqrt{6} + e^7 i\right| = \sqrt{6 + e^{14}}$$

You can check that their squares are, in fact obtained the same as the product of each number with its conjugate.

We are now ready to tackle a proper division of complex numbers.

If it were possible to compute the ratio $\frac{z_1}{z_2}$, with $z_2 \neq 0$, it would imply that:

$$\frac{z_1}{z_2} = \frac{z_1 \overline{z}_2}{z_2 \overline{z}_2} = \frac{1}{|z_2|^2} z_1 \overline{z}_2$$

But the expression on the right makes perfect sense, since it is obtained as the product of two complex numbers, a perfectly fine operation, combined with multiplication by a real scalar, also a perfectly good operation. So, we have a way!

Definition

Given two complex numbers z_1 and $z_2 \neq 0$, we define their ***quotient*** as:

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}.$$

Notice that what we are doing when dividing two complex numbers is, in effect, rationalizing the denominator!

Example: $\dfrac{2-3i}{5+i}$

By using our definition, we get:

$$\frac{2-3i}{5+i} = \frac{(2-3i)(5-i)}{(5+i)(5-i)} = \frac{7-17i}{26} = \frac{7}{26} - \frac{17}{26}i$$

But is this what we want? That is, if we multiply this "*quotient*" by the denominator, do we get the numerator? Try that: it is a good exercise!

Example: $\dfrac{\sqrt{6}+i}{2-\sqrt{3}i}$

Same idea and method, just slightly more messy computations.

$$\frac{\sqrt{6}+i}{2-\sqrt{3}i} = \frac{(\sqrt{6}+i)(2+\sqrt{3}i)}{(2-\sqrt{3}i)(2+\sqrt{3}i)} =$$

$$= \frac{2\sqrt{6}-\sqrt{3}+(2+3\sqrt{2})i}{7} = \frac{2\sqrt{6}-\sqrt{3}}{7} + \frac{2+3\sqrt{2}}{7}i$$

Let's stop now and allow you to become familiar with these basic concepts. In the next section we'll start to see a few ways in which complex numbers are related to the vectors we have been studying so far and to their properties.

Oh, and we may see the following symbol related to complex numbers.

Definition

The set of complex numbers is usually denoted by the symbol $\mathbb{C}$.

Summary

- Complex numbers are developed in order to allow work with the square root of negative numbers.
- The imaginary unit is defined as an entity denoted by *i* and whose square is -1.
- A complex numbers is a linear combination 1 and *i*.
- The basic algebraic operations of additions, subtraction and multiplication of are performed on complex numbers as we would for polynomials.
- Division of complex numbers is obtained by rationalizing through the conjugate of the denominator.
- Roots need to be handled carefully and will be studied at a later date.

Common errors to avoid

- Be careful when applying algebra to complex numbers. Although much of it applies, not all does and it is important to know where the tricky steps are.

Learning questions for Section LA 11-1: Basic definitions and operations

Review questions:

1. Explain what an imaginary number is.
2. Describe what a complex number is.
3. Compare and contrast the concepts of imaginary and complex numbers.
4. Describe how to perform addition, subtraction, multiplication and division between real, imaginary and complex numbers.

Memory questions:

1. What property identifies the imaginary unit *i*?
2. What formula identifies a complex number?
3. What is the conjugate of a complex number $a+bi$?
4. What is the modulus of a complex number $a+bi$?
5. How is the quotient of two complex numbers defined?

Computation questions:

Given the introductory nature of this section, the only questions involving computations are of a simple nature and the *Templated questions* offer you a way to get more of the same practice. Here are two questions that link to a basic concept not explicitly mentioned in the body of the section.

1. Determine the reciprocal of the complex number $3+4i$

2. Determine the reciprocal of the complex number $4-3i$.

Theory questions:

1. Does the definition of quotient of two complex numbers work as it does for real numbers when the denominator is a real number?

2. Explain why a complex number can be viewed as a linear combination of 1 and *i*.

3. Explain how to write a complex number as a dot product of two suitable vectors, one of which consists of the real and imaginary part of the number.

4. What is the reciprocal of i ?

Proof questions:

1. Prove that for any two complex numbers z_1, z_2 it is true that $|z_1 z_2| = |z_1||z_2|$.

2. Show that a formula for factoring a sum of two squares exists when allowing complex numbers.

3. Prove that for any 2D vector $[x \;\; y] \neq \mathbf{0}$, $\dfrac{x+yi}{y-xi} = i$.

Templated questions:

In these questions, make your own choice of any items involved.

1. For any two complex numbers z_1, z_2 of your choice:
   a) For each of them, compute their modulus, reciprocal and conjugate.
   b) Compute their sum, difference and product.
   c) Verify that the product of their moduli is the modulus of their product.
   d) Compute their quotient both ways.

What questions do you have for your instructor?