Chapter 11: Complex numbers **Section 2**

Complex numbers, vectors and calculus

What you need to know already:	What you can learn here:
➤ The definition and basic operations of complex numbers.	➤ How complex numbers can be viewed as vectors. ➤ What properties of complex numbers this perspective allows us to discover.

One of the learning questions of the previous section asked you to show that a complex number can be seen as a linear combination, while another asked you to show you that it can be written as a dot product.

This suggests that there may be some nice connections between complex numbers and vectors.

I am not surprised, since you are talking about them in a linear algebra course!

Yes, although it is also common to introduce complex numbers within the context of calculus, advanced algebra or other mathematical subjects. But let us explore this connection with vectors a little more.

Technical fact

A complex number $z = a + bi$ can be ***represented*** as a 2D vector, whose first component is the real part of z and whose second component is its imaginary part:

$$z = a + bi \quad \leftrightarrow \quad \begin{bmatrix} a & b \end{bmatrix}$$

With this representation:

- ➤ addition of complex numbers can be performed as vector addition;
- ➤ multiplication by a real number can be performed by means of scalar multiplication;
- ➤ The modulus of z is the length of the corresponding vector.

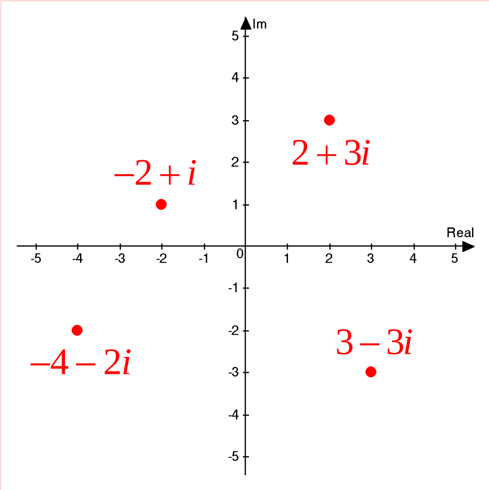
What about other connections between properties of complex numbers and of vectors?

In general, they tend to be a little trickier, but we'll see some of them in the rest of the section, or it would be a very short one!

First, the connection between complex numbers and 2D vectors suggests that a complex number can be represented as a point in the Cartesian plane. This leads to a standard way to represent complex numbers.

Technical fact

Complex numbers can be visually represented in the Cartesian plane by considering the ***horizontal axis*** as representing the ***real part*** of the number and the ***vertical axis*** as the ***imaginary part***.



This representation does not give us just a visual aid, but many ideas as well. The first important one is that we can identify a complex number by using the polar coordinates of the corresponding point.

Definition

The ***polar form*** of a complex number $z = a + bi$ is obtained by writing it in terms of its corresponding polar coordinates:

$$a + bi = r(\cos\theta + i\sin\theta)$$

In this form, $r = |z| = \sqrt{a^2 + b^2}$ and θ, called the ***argument*** of z, is the smallest non-negative value that we can use for the usual polar form or an equivalent value obtained from it by adding a multiple of 2π .

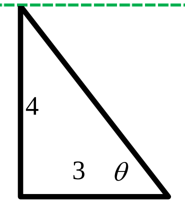
Example: $z_1 = 1 + i,\ z_2 = 3 - 4i$

The number $z_1 = 1 + i$ corresponds to the point $(1,1)$, whose polar coordinates are $\left(\sqrt{2}, \frac{\pi}{4}\right)$. Therefore, $z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$. In fact:

$$z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1 + i$$

Similarly, the number $z_2 = 3 - 4i$ corresponds to a point in the fourth quadrant, whose polar coordinates may be written as $\left(5, -\tan^{-1}\frac{4}{3}\right)$.

Therefore, its polar form is $5\left(\cos\left(-\tan^{-1}\frac{4}{3}\right) + i\sin\left(-\tan^{-1}\frac{4}{3}\right)\right)$. In fact, by using a triangle model, as for inverse trigonometric substitution for integrals, we have:



$$z_2 = 5(\cos\theta - i\sin\theta) = 5\left(\frac{3}{5} - \frac{4}{5}i\right)$$

Why the complication with θ? Why not simply use the usual $\theta = \tan^{-1}\dfrac{b}{a}$ *?*

As you know from your work with polar coordinates, in that context we allow the value of *r* to be negative, but here we need *r* to be a positive real number, for reasons that we'll see shortly. This will allow us to properly use this form in the context of the product of complex numbers. It will not prove to be a big deal, though.

So, what about the product of complex numbers? How does that relate?

Here is the scoop, compared also to the more familiar addition.

Technical fact

Two complex numbers can be added by considering them as vectors and using vector addition:

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

Two complex numbers can be multiplied by relying on usual algebra in their common form:

$$(a+bi)(c+di)=(ac-bd)+(ad+bc)i$$

They can also be multiplied by using determinants:

They can also be multiplied by using determinants:

$$(a+ib)(c+id)=\begin{vmatrix} a & b \\ d & c \end{vmatrix} + i\begin{vmatrix} a & b \\ -c & d \end{vmatrix}$$

But they can most easily be multiplied by using their polar form, since:

$$\left[r_1(\cos\theta_1 + i\sin\theta_1)\right]\times\left[r_2(\cos\theta_2 + i\sin\theta_2)\right] =$$
$$= r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$$

Proof

The first two claims were proven in the previous section and involve just basic algebra.

The claim about the determinant based formula can be verified easily: a little exercise for you!

The last claim is the most interesting, as it implies that the product is indeed most easily done in polar form. That the modulus part is the product of the two moduli is clear from the algebra and the fact that the modulus of a product is the product of the two moduli (see previous section). As for the angle part, we have:

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) =$$
$$= \cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$
$$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$$

But the real and imaginary parts are given by two addition formulae for trig functions, and they lead to the expression:

$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$

as claimed.

Example: $z_1 = 1+i,\ z_2 = 3-4i$

I let you multiply these two numbers by using the *FOIL* method. By using determinants we have:

$$(1+i)(3-4i) = \begin{vmatrix} 1 & 1 \\ -4 & 3 \end{vmatrix} + i\begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = 7 - i$$

By using their polar form, we have:

$$(1+i)(3-4i) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)5\left(\cos\left(-\tan^{-1}\frac{4}{3}\right) + i\sin\left(-\tan^{-1}\frac{4}{3}\right)\right)$$
$$= 5\sqrt{2}\left(\cos\left(\frac{\pi}{4} - \tan^{-1}\frac{4}{3}\right) + i\sin\left(\frac{\pi}{4} - \tan^{-1}\frac{4}{3}\right)\right)$$

That last one does not seem very easy.

True, but that is because we wrote the angle in exact numerical form. If we used identifying symbols (θ_1, θ_2) or approximate numerical values, it would be easier. The former method is used in theoretical work or intermediate steps, while the latter is used in actual computations.

Notice that this states that to multiply two complex numbers in polar form, we multiply their modulus and add their angle. And if we can multiply this way we can also divide.

Technical fact

The quotient of two complex numbers in polar form is given by:

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)$$

Proof

If you multiply the given form of the quotient by the denominator, by using the multiplication rule, you do get the numerator. Another exercise for your pleasurable practice!

Example: $z_1 = 1+i,\ z_2 = 3-4i$

By dividing with the conjugate method we get:

$$\frac{1+i}{3-4i} = \frac{1+i}{3-4i}\,\frac{3+4i}{3+4i} = -\frac{1}{25} + \frac{7}{25}i$$

By using their polar form, we have:

$$\frac{1+i}{3-4i} = \frac{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}{5\left(\cos\left(-\tan^{-1}\frac{4}{3}\right) + i\sin\left(-\tan^{-1}\frac{4}{3}\right)\right)}$$
$$= \frac{\sqrt{2}}{5}\left(\cos\left(\frac{\pi}{4} + \tan^{-1}\frac{4}{3}\right) + i\sin\left(\frac{\pi}{4} + \tan^{-1}\frac{4}{3}\right)\right)$$

Arguably a longer expression, but easier steps.

And what else can you do once you know how to multiply?

Compute powers and roots!

Technical fact

The following formulae are valid:

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$$
$$\left[r(\cos\theta + i\sin\theta)\right]^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

Proof

All you need to do is apply the product formula for the power and reverse the power formula for the root.

Example: $z_1 = 1+i,\ z_2 = 3-4i$

This time we easily get:

$$(1+i)^2 = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^2 = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2i$$

$$(3-4i)^3 = \left[5\left(\cos\left(-\tan^{-1}\frac{4}{3}\right) + i\sin\left(-\tan^{-1}\frac{4}{3}\right)\right)\right]^3$$

$$= 125\left(\cos\left(-3\tan^{-1}\frac{4}{3}\right) + i\sin\left(-3\tan^{-1}\frac{4}{3}\right)\right)$$

But we need to pay attention to one technical detail. Remember that there are infinitely many values we can use for θ, all differing by a multiple of 2π. This means that there are several possible roots: a problem we found earlier, in the definition of *i*. But mathematicians have learned how to handle this issue and even how to extract some more interesting information from it. Investigating the roots of a complex number, and especially the complex roots of 1, is an interesting area, but beyond our current goals. I hope you are curious enough to take more courses on complex numbers: they are fascinating! And those courses may even be required!

But, having connected to vectors, polar coordinates and trigonometric functions, we'll finish with an important connection to calculus.

Technical fact

Any complex number whose ***modulus is 1*** can be written as $e^{i\theta}$ for some value of θ. That is:

$$|z| = 1 \quad \Rightarrow \quad z = e^{i\theta} = \cos\theta + i\sin\theta$$

Proof

A complex number whose modulus is 1 can be written as:

$$z = \cos\theta + i\sin\theta$$

We know from calculus (and by now you should have seen it there) that:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

We also know that this series is absolutely convergent for all values of *x*, so that the order of its terms can be changed at will. So, we evaluate this function and the series for $x = i\theta$ and get:

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\right)$$

But the first series provides $\cos\theta$ and the second provides $\sin\theta$. Hence the conclusion.

Wow! That's a very simple proof for such a surprising fact!

Well, a mathematician might complain that this "proof" is missing a few details. But they are just that, technical details, while the essence of the proof is all there. Think of this as a demonstration of the power of complex numbers to explain and expand on known facts. Notice that we can now state the following.

Technical fact

Every complex number can be written as $re^{i\theta}$, also called it ***exponential form***. Moreover:

$$\left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

$$\left(re^{i\theta}\right)^k = r^k e^{ik\theta}$$

Example: $z_1 = 1+i,\ z_3 = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

We can first of all rewrite these numbers as:

$$z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}e^{i\frac{\pi}{4}}$$

$$z_3 = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 3e^{i\frac{\pi}{3}}$$

Then we can write, for instance:

$$z_1 z_3 = 3\sqrt{2}e^{i\frac{7\pi}{12}} \qquad \frac{z_1}{z_3} = \frac{\sqrt{2}}{3}e^{-i\frac{\pi}{12}}$$

$$z_1^2 = 2e^{i\frac{\pi}{2}} \qquad z_3^3 = 27e^{i\pi} = -27$$

Wait! But isn't -3 the only cube root of -27?

If you stick to real numbers, yes! But if you allow complex numbers, you get other roots! In fact, if we allow complex numbers, it turns out that a polynomial equation of degree *n* has exactly *n* roots, although some of them may have higher multiplicity. So, the equation $z^3 = -27$ has three roots. We found two, can you get the third?

And let me conclude this very quick introduction to amazing facts with a very famous equation.

Technical fact

$$e^{i\pi} + 1 = 0$$

This formula is commonly known as Euler's formula (although Euler was not the first to notice or prove it), or as the most beautiful mathematical equation.

Most beautiful? Why?

Because in its shortness, it manages to use the two most basic real numbers (0 and 1), the three most basic and important irrational numbers (e, π and i), the two most basic operations (addition and multiplication) and the most popular mathematical symbol (=). Can you think of another equation that can rival it in this way?

And here is its proof, which at this point is really easy!

Proof

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \Rightarrow \quad e^{i\pi} = \cos\pi + i\sin\pi$$

$$\Rightarrow \quad e^{i\pi} = -1 \quad \Rightarrow \quad e^{i\pi} + 1 = 0$$

Summary

- Complex numbers may be written in terms of the polar coordinates of the corresponding point in the Cartesian plane.
- The polar form can be used effectively to perform algebraic operation in an efficient way.
- Thanks to an important fact about infinite series, complex numbers may also be written in an even shorter exponential form, which also provides efficient and effective ways to perform algebraic operations.
- The exponential form of a complex number leads to the beautiful equation $e^{i\pi} + 1 = 0$, commonly considered as a gem of mathematical knowledge.

Common errors to avoid

- Not much, besides errors coming from other computational misdeeds.
- Learn how to use the different form in their proper manner and do not mix additions and multiplications.
- Remember than in the polar form of a complex number, the modulus needs always be positive.

Learning questions for Section LA 11-2: Complex numbers, vectors and calculus

Review questions:

1. Describe how complex numbers are represented in the Cartesian plane.
2. Explain how the polar form of a complex number is obtained.
3. Describe three methods for multiplying complex numbers.
4. Explain how the polar and exponential form of a complex number are related.

Memory questions:

1. What is the modulus of a complex number?
2. What is the argument of a complex number?

3. What does the polar form of a complex number look like?

4. What does the exponential form of a complex number look like?

5. Which formula provides the product of two complex numbers in their polar form?

6. Which formula provides the product of two complex numbers in their exponential form?

7. Which formula provides the quotient of two complex numbers in their polar form?

8. Which formula provides the quotient of two complex numbers in their exponential form?

9. Which formula provides the power of a complex number in its polar form?

10. Which formula provides the power of a complex number in its exponential form?

Computation questions:

1. Use the polar form of a complex number to compute $\ln i$.

2. Use the polar form of a complex number to compute $(-1)^{\pi}$.

3. Compute the polar and standard forms of the complex number whose exponential form is e^{i}.

4. Obtain the polar and standard form of the complex number $\left(e^{i}\right)^{2}$.

5. Using the fact that $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ (see proof question later) find a value of *z* for which $\sin z = 2$.

6. Identify all three complex roots of the equation $z^{3} = -27$.

Proof questions:

1. Prove that $(a+ib)(c+id) = \begin{vmatrix} a & b \\ d & c \end{vmatrix} + i \begin{vmatrix} a & b \\ -c & d \end{vmatrix}$.

2. Prove that $\dfrac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \dfrac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)$.

3. Mimic the proof of the formula $e^{i\theta} = \cos\theta + i\sin\theta$ to show that for any complex number *z*, $e^{iz} = \cos z + i\sin z$.

4. Use the exponential form of a complex number of modulus 1 to show that $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$. (HINT: Use the result of the previous question for both *z* and –*z*).

5. For any positive real number *x*, determine the value of x^{i}.

Templated questions:

In these questions, make your own choice of any items involved.

1. Choose two fairly simple complex numbers and compute their polar and exponential form.

2. Choose two fairly simple complex numbers and compute their product, quotient and cube in both polar and exponential form.

What questions do you have for your instructor?