Some examples of vector spaces

What you need to know already:

- The ten axioms needed to identify a vector space.

What you can learn here:

- Some examples of vector spaces related to, but also different from the Euclidean ones seen so far.

The whole idea of vector spaces started with Euclidean vectors in mind, so it is not surprising that, as you discovered in the previous section, $\mathbb{R}^n$ is a vector space for any *n*. But it should also not be surprising that there are any other sets that are also vector spaces. We shall make their acquaintance gradually, starting, in this section, with sets that still consist of Euclidean vectors.

But before going there, I want to introduce a strange fellow, a vector space that we shall meet often, despite, or in fact *because* of its simplicity.

Definition

A set consisting of a single object, denoted by $\mathbf{0}$, with operations defined by $\mathbf{0}+\mathbf{0}=\mathbf{0}$ and $c\mathbf{0}=\mathbf{0}$ is called the ***trivial vector space***.

Technical fact

The ***trivial vector space*** satisfies all vector space axioms and is therefore truly a vector space.

The proof of this fact is very simple, although it may create some head-scratching because of its simplicity. I encourage you to check its truth by completing the corresponding *Learning question*.

So, in particular, the zero vector of any $\mathbb{R}^n$, with the usual operations, is an instance of the trivial vector space. But so am I, so long as I use the operations:

$$me \oplus me = me \qquad c \otimes me = me$$

Are you calling "me" a trivial vector space? ☺

Yes, but it is not an offensive term! If the idea is still confusing, file it away for now and wait until we put it to good use. Let us look at a less trivial example.

Technical fact

The set of vectors in $\mathbb{R}^n$ ***whose components add up to 0***, together with the usual operations, forms a ***vector space***.

Proof

Since we are using the usual operations and we know that for them all algebraic and special item axioms work, we only need to check the closure axioms. Also, by what we have seen, it is enough to check the *LC* axiom.

So, if **u** and **v** are vectors in $\mathbb{R}^n$ such that $\sum_{i=1}^{n} u_i = 0$ and $\sum_{i=1}^{n} v_i = 0$, and if *c* and *d* are two scalars, then:

$$c\mathbf{u} + d\mathbf{v} = \left[cu_1 + dv_1 \quad cu_2 + dv_2 \quad \cdots \quad cu \quad + dv_n \right]$$

This implies that:

$$\sum_{i=1}^{n} (c\mathbf{u} + d\mathbf{v})_i = \sum_{i=1}^{n} (cu_i + dy_i) = \sum_{i=1}^{n} cu_i + \sum_{i=1}^{n} dy_i$$

$$= c\sum_{i=1}^{n} u_i + d\sum_{i=1}^{n} y_i = c0 + d0 = 0$$

Therefore the linear combination is also in the set and the *LC* axiom works.

Notice that the key fact we used in the proof is the right side of the equation being 0, so that we can generalize this fact.

Technical fact

The set of ***solutions of a homogeneous linear equation*** in *n* variables, with the same operations as in $\mathbb{R}^n$, forms a ***vector space***.

Proof

Once again, all we need to check is the *LC* axiom.

Let's say that $a_1x_1 + a_2x_2 + \ldots + a_nx_n = \mathbf{a} \bullet \mathbf{x} = 0$ is our homogeneous equation, that **u** and **v** are two solutions for it and that *c* and *d* are any two scalars. To check that $c\mathbf{u} + d\mathbf{v}$ is also a solution we only need to use it as **x** in the equation and follow usual algebra:

$$a_1(cu_1 + dv_1) + a_2(cu_2 + dv_2) + \ldots + a_n(cu_n + dv_n) =$$

$$= a_1cu_1 + a_1dv_1 + a_2cu_2 + a_2dv_2 + \ldots + a_ncu_n + a_ndv_n =$$

$$= (ca_1u_1 + ca_2u_2 + \ldots + ca_nu_n)$$

$$+ (da_1v_1 + da_2v_2 + \ldots + da_nv_n) =$$

$$= c(a_1u_1 + a_2u_2 + \ldots + a_nu_n) + d(a_1v_1 + a_2v_2 + \ldots + a_nv_n)$$

But since both **u** and **v** are solutions, the quantities in brackets are both 0, hence so is the entire expression and the linear combination is also a solution.

But we also know that a homogeneous equation in *n* variables describes a hyperplane in $\mathbb{R}^n$ that contains the origin. Since we are dealing with full-force generalizations, here is a characterization of *all* vector spaces that consist of Euclidean vectors with the usual operations.

Technical fact

A subset of $\mathbb{R}^n$ is a ***subspace*** if and only if it is a ***vector space with the usual operations***.

Proof

If it is a vector space with the usual operations, the closure axioms hold and the set is closed under linear combinations. That means that it is a subspace.

If it is a subspace, the closure axioms must hold for the same operations. Since all other axioms hold because we are dealing with the same operations, it is a vector space.

What exactly did we prove? It looks like you just said the same thing in different ways!

I did! But remember what I said about looking at the same thing from different perspectives: I was just checking that the object was the same, despite the different point of view and terminology used. Also, remember that we shall soon apply these concepts to more general vector spaces, so that the fine points of logic must be carefully checked each time, even if they look kind of obvious.

So, back to hyperplanes as a special case of the last fact.

Technical fact

The set of points that make up any ***hyperplane*** of $\mathbb{R}^n$ containing the origin, with the usual operations, forms a ***vector space***.

Example: $3x - 2y = 0$

This is a hyperplane of $\mathbb{R}^n$ for any $n \geq 2$ (generalization!) and contains the origin and therefore it is a vector space.

Notice that the origin must be in the hyperplane, or we do not get a vector space with the usual operations. That will change soon!

Example: $x + 3y - 2z = 4$

This plane is a hyperplane of $\mathbb{R}^3$, but it does not form a vector space with the usual operations, as it is not closed under addition or scalar multiplication. To see this all we need is one counterexample, so here it is.

The point $(0, 2, 1)$ is on the plane, since $0+3(2)-2(1)=6-2=4$.

However, its double, $(0, 4, 2)$, as a vector, is not on the plane, since $0+3(4)-2(2)=12-4=8\neq 4$. One counterexample is enough.

But a hyperplane that does not contain the origin still looks like one that does, so why is it not a vector space?

Good point: notice that I said that it does not form a vector space *with the usual operations*! Remember that a vector space consists of both a set of objects *and* two operations.

We can fix things by changing operations, but in that case we need to check the rest of the axioms. This is best done by using the vector equation of a hyperplane. I will state and show you this in $\mathbb{R}^3$, since the general case only requires the use of ellipsis … (not ellipses!)

What???!!!

Don't believe me? Pay attention to the proof!

Technical fact

The vectors on the ***hyperplane*** of $\mathbb{R}^3$ defined by $\mathbf{x} = \mathbf{p} + h\mathbf{a} + k\mathbf{b}$ form a ***vector space with the following operations***:

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v} - \mathbf{p}$$
$$c \otimes \mathbf{u} = c\mathbf{u} + (1 - c)\mathbf{p}$$

Proof

In the following, let us assume that $\mathbf{u} = \mathbf{p} + h_1\mathbf{a} + k_1\mathbf{b}$, $\mathbf{v} = \mathbf{p} + h_2\mathbf{a} + k_2\mathbf{b}$ and that *c* and *d* are scalars. Then the closure axioms work because:

C1) $\mathbf{u} \oplus \mathbf{v} = (\mathbf{p} + h_1\mathbf{a} + k_1\mathbf{b}) + (\mathbf{p} + h_2\mathbf{a} + k_2\mathbf{b}) - \mathbf{p}$

$= \mathbf{p} + (h_1 + h_2)\mathbf{a} + (k_1 + k_2)\mathbf{b}$

C2) $c \otimes \mathbf{u} = c(\mathbf{p} + h_1\mathbf{a} + k_1\mathbf{b}) + (1 - c)\mathbf{p} = \mathbf{p} + h_1\mathbf{a} + k_1\mathbf{b}$

As for the algebraic axioms, A1 follows from usual commutativity of vector addition and A2 works because:

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = \mathbf{u} \oplus (\mathbf{v} + \mathbf{w} - \mathbf{p}) = \mathbf{u} + \mathbf{v} + \mathbf{w} - \mathbf{p} - \mathbf{p}$$
$$= (\mathbf{u} + \mathbf{v} - \mathbf{p}) + \mathbf{w} - \mathbf{p} = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$$

I leave to you the task to check the other algebraic axioms.

The vector **p** plays the role of the **0** vector, since:

$$\mathbf{u} \oplus \mathbf{p} = \mathbf{u} + \mathbf{p} - \mathbf{p} = \mathbf{u}$$

Also, the negative of **u** will be $2\mathbf{p} - \mathbf{u}$, since:

$$\mathbf{u} \oplus (2\mathbf{p} - \mathbf{u}) = \mathbf{u} + 2\mathbf{p} - \mathbf{u} - \mathbf{p} = \mathbf{p}$$

and this is the zero vector, by our last verification.

I also leave to you the task of checking S3.

This is crazy! What does it all mean?

Another excellent question! If you reflect on what these weird operations do, you will realize that they take the original hyperplane, move it enough so as to incorporate the origin (by subtracting **p**), perform the usual operations on the new hyperplane that contains the origin, and finally move everything back to the original hyperplane!

Of course we can continue to play this game in all kinds of directions, but I will not take it too far for now, except to show you one further generalization that will come in handy later.

Technical fact

The set of ***solutions of any homogeneous system*** $\mathbf{Ax} = \mathbf{0}$ forms a ***vector space***.

The set of ***solutions of a consistent non-homogeneous system*** $\mathbf{Ax} = \mathbf{c}$ can be obtained by picking one solution and adding to it every vector that is a solution of the ***associated homogeneous*** system $\mathbf{Ax} = \mathbf{0}$.

I bet you want me to check this too!

Yes, since it will help you build familiarity with this strange new concept of different operations. But let me help you with that by showing you the details of another, even weirder example. But beware: we shall get far from the familiar territory of algebra only to realize that we are actually still there!

Technical fact

The set $\mathfrak{L}$ of positive real numbers with the following two operations forms a vector space:

- ***Addition***: For any two numbers x and y in $\mathfrak{L}$, $x \oplus y = xy$, where the product on the right is the usual one.
- ***Scalar multiplication***: For any number x in $\mathfrak{L}$ and any scalar c, $c \otimes x = x^c$, where the power on the right side is the usual one.

Proof

So, we are considering the set of positive real numbers, but by *addition* we now really mean multiplication and by *scalar multiplication* we really mean exponentiation. Let's see if the axioms all hold.

To help you along I will use **bold** to refer to one of the weird operations above and to identify a positive number when viewed as a vector in this weird space.

Axiom 1: The **addition** of two positive real numbers is indeed a positive real number, since they are being multiplied together.

Axioms 2, 3: Commutativity and associativity of **addition** hold since it holds for the usual product of numbers.

Axiom 4: Here we may seem to run into trouble, since the usual number 0 is not even in our set (remember that we are dealing with *positive* real numbers). But the axiom just requires the existence of one vector with the property of **0**, it does not have to be the number 0. In fact, the positive real number **1** plays the role of the zero vector here. Look:

$$\mathbf{x} \oplus \mathbf{1} = \mathbf{1} \oplus \mathbf{x} = 1x = x = \mathbf{x}$$

And this is what we want, isn't it?

Axiom 5: Again, we do not have negative numbers, but we do not need them either. All we need is a positive number that fulfills the axiom for the weird operations we have defined. And it so happens that reciprocals do the job nicely: for any positive number **x** if **y** is its reciprocal as a usual number, then

$$\mathbf{x} \oplus \mathbf{y} = \mathbf{x} \oplus \frac{1}{\mathbf{x}} = x\frac{1}{x} = \mathbf{1}$$

And didn't we just see that the number **1** plays the role of the 0 vector? So we are fine here too. (I realize that the interplay of usual numbers and vectors in this vector space of numbers is confusing the first time around, so you may want to look at this proof again).

Axiom 6: For any positive number **x** and any scalar c we have that $c \otimes \mathbf{x} = x^c$, which is a positive number whatever c is, so this closure axiom holds.

Axiom 7: Same notation as before, try to follow the logic:

$$(cd) \otimes \mathbf{x} = x^{cd} = x^{dc} = \left(x^d\right)^c = c \otimes \left(x^d\right) = c \otimes (d \otimes \mathbf{x})$$

Axiom 8: Words should not be needed any more: $1 \otimes \mathbf{x} = x^1 = \mathbf{x}$

Axiom 9: $c \otimes (\mathbf{x} \oplus \mathbf{y}) = (xy)^c = x^c y^c = (c \otimes \mathbf{x}) \oplus (c \otimes \mathbf{y})$

Axiom 10: Oh, I am sure you can check this one yourself, so I will not deny you such pleasure.

As you puzzle over this really strange example, it may help you to consider the fact that you have seen these strange operations before. Does the word "*logarithm*" ring any bell?

Time for a pause, some more practice and maybe a pleasurable drink before moving on to meeting other vector spaces that are more familiar, while looking less as usual vectors.

Summary

- Yes, Virginia, there are vector spaces that are not made up of Euclidean vectors, but many of them still are!

Common errors to avoid

- Avoid not practicing! This is a new area of math and if you don't play and struggle with it, it will never become familiar!

Learning questions for Section LA 11-2

The *Learning questions* for this section are more streamlined than usual, focusing on exercises to increase your familiarity with the vector space concept. Regular programming will resume shortly.

Proof questions:

For each the sets described in questions 1-6, determine whether the set, with the usual operation, is or is not a vector space. To determine that it is, show that all axioms are satisfied; to determine that it is not, identify at least one axiom that fails.

1. The trivial vector space.
2. The set of all points on the unit circle of $\mathbb{R}^2$ with usual operations.
3. The set of 3D vectors $\mathbf{v}=\begin{bmatrix} x & y & z \end{bmatrix}$ that satisfy an equation of the type $ax+by+cz=0$.
4. The set of 3D vectors $\mathbf{v}=\begin{bmatrix} x & y & z \end{bmatrix}$ that satisfy an equation of the type $ax+by+cz=1$.
5. The set of all geometric vectors $\begin{bmatrix} x & y \end{bmatrix}$ in $\mathbb{R}^2$ that satisfy the equation $y=x^2$.
6. The set of solutions of any homogeneous system $\mathbf{Ax}=\mathbf{0}$

Determine whether each of the sets described in questions 7-12 forms a vector space with respect to the indicated operations. To determine that it is, show that all axioms are satisfied; to determine that it is not, identify at least one axiom that fails.

7. The set of four dimensional Euclidean vectors, with usual scalar multiplication and dot product as addition.

8. The set of usual 3-dimensional vectors, with the usual addition, but with scalar multiplication defined by $k(x, y, z) = (x^k, y^k, z^k)$

9. The set of usual 3-dimensional vectors, with the usual addition, but with scalar multiplication defined by $k(x, y, z) = (e^k x, e^k y, e^k z)$

10. The set of three dimensional Euclidean vectors with usual addition and scalar multiplication given by $k\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} a^k & b^k & c^k \end{bmatrix}$.

11. The set of usual 3-dimensional vectors, with the usual scalar multiplication, but with addition defined by $(x, y, z) + (u, v, w) = (x-u, y-v, z-w)$.

12. The set of usual 3-dimensional vectors, with the usual scalar multiplication, but with addition defined by $(x, y, z) + (u, v, w) = \left(\frac{x+u}{2}, \frac{y+v}{2}, \frac{z+w}{2}\right)$.

13. Prove that the points on a hyperplane of $\mathbb{R}^n$ not containing the origin do form a vector space by using suitably define operations.

14. Consider the set consisting of all real multiples of the variable *x*, all real multiples of the variable *y* and an additional object denoted by $\perp$. That is, our set consists of $\{cx, dy, \perp\}$, where *c* and *d* represent any real numbers. Also, consider the operations of addition and scalar multiplication defined by:

$$\text{Addition } \oplus : \begin{cases} ax \oplus bx = (a+b)x \\ cy \oplus dy = (c+d)y \\ ax \oplus cy = cy \oplus ax = \perp \\ ax \oplus \perp = \perp \oplus ax = \perp \\ cy \oplus \perp = \perp \oplus cy = \perp \\ \perp \oplus \perp = \perp \end{cases}$$

$$\text{Scalar multiplication } \otimes : \begin{cases} k \otimes ax = (ka)x \\ k \otimes cy = (kc)y \\ k \otimes \perp = \perp \end{cases}$$

Prove that this set, with these operations, satisfies eight of the ten axioms of a vector space. That is, prove that eight of the ten axioms are true and show why the remaining axioms do not work. Of course, you will have to identify which axioms do not work!

What questions do you have for your instructor?