Chapter 11: Complex numbers — **Section 3**

Linear algebra with complex numbers

What you need to know already:	What you can learn here:
➤ What complex numbers are. ➤ How to perform algebraic operations on complex numbers.	➤ How we can extend linear algebra methods to the situation when complex numbers are used as scalars.

Throughout the course, we have used real numbers as scalars. We use them to define vectors, matrices and even to define complex numbers. But now that we have these new numbers, on which we can do all basic operations of linear algebra, we can ask whether in fact we extend all of linear algebra to include them. The answer is an unsurprising, but exciting yes!

Technical fact

By using $\mathbb{C}$ as the basic set of scalars, we can extend all basic definitions and of linear algebra to the complex numbers.

In particular, in a manner fully analogous to what we did with real numbers, we can define:

- The set $\mathbb{C}^n$ of n-dimensional vectors with complex components.
- The set of $n \times m$ matrices with complex entries.
- The concepts of vector addition, scalar multiplication, dot and matrix product.
- The concepts of subspace, span and basis.
- The concept of orthogonality, although this does not have the same geometric meaning that we are familiar with. Similarly, the related concept of angle must be adapted to the algebraic nature of the setting.
- The concept of eigenvalues and eigenvectors.

So, in other words, whatever we can do with real numbers, we can also do with complex numbers. Only better, deeper and more interestingly, I might add!

In fact, there are some other things we can do and are not listed above. Can you think of some of them?

Because of this wonderful extension, we may need to specify whether we are dealing with complex numbers, or real numbers only. The former case is usually indicated in one of these ways.

Definition

When dealing with linear algebra tools and concepts by allowing complex numbers as scalars, we can say that we are ***working with complex scalars***, or that we are ***using complex components and entries***, or that we ***work over*** $\mathbb{C}$.

Wow! We could do a whole new course on that!

Indeed more than one! And in fact this is an area of mathematics that is very reach and has lots of unexpected practical uses, despite the initial feeling that imaginary numbers can only lead to imaginary consequences. But they are the topic of, as you said, another course, which I sincerely hope you will take!

For now, and to conclude this course, let me give you a few examples of how all this is done and of some very basic facts related to it. Notice, as you go through them, that while some properties remain pretty much unchanged when using complex scalars, some others do change. For instance, the concept of angle must be redefined, as noted above, since by using our definition we end up with the cosine of a complex number and this does not identify an angle in the usual sense.

Example: $\begin{bmatrix} i & 1 \\ 2 & -i \end{bmatrix}$

This is a 2×2 matrix with complex coefficients. Its rows are the complex vectors $[i \;\; 1], [2 \;\; -i]$, which are vectors in $\mathbb{C}^2$.

Its row space consists of the span of these two vectors, that is, it consists of all vectors that can be written as linear combinations of these two:

$$Span\{[i \;\; 1],[2 \;\; -i]\} = \{[iz_1 \;\; z_1]+[2z_2 \;\; -iz_2]\} = \{[iz_1+2z_2 \;\; z_1-iz_2]\}$$

But the same situation within real numbers would tell us that this span includes all 2D vectors: is that true here as well? Indeed it is, but it needs verification: look for it in the learning questions!

And this being a square matrix, we can compute its determinant:

$$\left\|\begin{bmatrix} i & 1 \\ 2 & -i \end{bmatrix}\right\| = i(-i)-2=-1$$

This time the determinant is real, but of course it needs not be: can you construct a matrix with a complex determinant?

And since the determinant is not 0, the matrix should be invertible, right? Well, let's find its inverse by the usual Gauss-Jordan method. Of course, here we can use complex numbers as scalars!

$$\left[\begin{array}{cc|cc} i & 1 & 1 & 0 \\ 2 & -i & 0 & 1 \end{array}\right] \underset{\Rightarrow}{-i\mathbf{R}_1} \left[\begin{array}{cc|cc} 1 & -i & -i & 0 \\ 2 & -i & 0 & 1 \end{array}\right] \underset{\Rightarrow}{\mathbf{R}_2-2\mathbf{R}_1} \left[\begin{array}{cc|cc} 1 & -i & -i & 0 \\ 0 & i & 2i & 1 \end{array}\right]$$

$$\underset{\Rightarrow}{\mathbf{R}_1+\mathbf{R}_2} \left[\begin{array}{cc|cc} 1 & 0 & i & 1 \\ 0 & i & 2i & 1 \end{array}\right] \underset{\Rightarrow}{-i\mathbf{R}_2} \left[\begin{array}{cc|cc} 1 & 0 & i & 1 \\ 0 & 1 & 2 & -i \end{array}\right]$$

Is that correct? Is this matrix its own inverse?

$$\begin{bmatrix} i & 1 \\ 2 & -i \end{bmatrix}\begin{bmatrix} i & 1 \\ 2 & -i \end{bmatrix} = \begin{bmatrix} -1+2 & i-i \\ 2i-2i & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Yes! But we have seen that a matrix can be its own inverse even among those with real entries, so it is not a surprise.

And while we are at it, let's find the eigenvectors and eigenvalues of this matrix.

$$\left\|\begin{bmatrix} i-\lambda & 1 \\ 2 & -i-\lambda \end{bmatrix}\right\| = (i-\lambda)(-i-\lambda)-2 = 1+\lambda^2-2=\lambda^2-1$$

So, the eigenvalues are $\lambda=\pm 1$, both being real. That's nice, not surprising, but not universally true: eigenvalues can be complex if the matrix has complex entries. I leave to it the finding of the eigenspaces.

So, this example shows that determinant end eigenvalues of a complex matrix can be real, but need not be so. However, the following fact is very important and you will likely encounter it again in future course. I think that that's a great way to end the course: don't you?

Technical fact

Every $n \times n$ matrix over $\mathbb{C}$ has ***exactly n eigenvalues***, as long as we count each of them with its algebraic multiplicity. However, their geometric multiplicities may not add up to *n*.

Proof

The eigenvalues of an $n \times n$ matrix are the roots of its characteristic polynomial. According to the Fundamental Theorem of Algebra, a polynomial of degree *n* with complex coefficients has exactly *n* roots, when counted with their algebraic multiplicities. This is the claim of the first statement.

As for the second statement, we know this to be true when using real numbers and real numbers are also complex numbers. So, any counterexample with real eigenvalues is sufficient evidence for the claim.

Example: $\begin{bmatrix} 1 & i & -i \\ 2i & 0 & 3 \\ i & 1 & 2 \end{bmatrix}$

We know for sure that the algebraic multiplicities of the eigenvalues of this matrix add up to 3. But do we have 3 eigenvalues or less? Let's find out.

$$\begin{vmatrix} 1-\lambda & i & -i \\ 2i & -\lambda & 3 \\ i & 1 & 2-\lambda \end{vmatrix} \overset{\mathbf{R}_2 - 2\mathbf{R}_3}{=} \begin{vmatrix} 1-\lambda & i & -i \\ 0 & -\lambda-2 & 2\lambda-1 \\ i & 1 & 2-\lambda \end{vmatrix} =$$

$$= (1-\lambda)\begin{vmatrix} -\lambda-2 & 2\lambda-1 \\ 1 & 2-\lambda \end{vmatrix} + i\begin{vmatrix} i & -i \\ -\lambda-2 & 2\lambda-1 \end{vmatrix}$$

$$= (1-\lambda)\left[(-\lambda-2)(2-\lambda)-(2\lambda-1)\right] + i\left[i(2\lambda-1)-i(\lambda+2)\right]$$

$$= (1-\lambda)\left[\lambda^2 - 2\lambda - 3\right] + i\left[\lambda i - 3i\right] = -\lambda^3 + 3\lambda^2 = \lambda^2(3-\lambda)$$

Therefore, $\lambda = 0$ is an eigenvalue with algebraic multiplicity 2 and $\lambda = 3$ is an eigenvalue with algebraic, and hence geometric multiplicity 1.

Let's find the eigenspace for $\lambda = 0$:

$$\begin{bmatrix} 1 & i & -i \\ 2i & 0 & 3 \\ i & 1 & 2 \end{bmatrix} \begin{matrix} \mathbf{R}_2 - 2i\mathbf{R}_1 \\ \mathbf{R}_3 - i\mathbf{R}_1 \\ \Rightarrow \end{matrix} \begin{bmatrix} 1 & i & -i \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

We can already see that the geometric multiplicity here is 1. The eigenvectors are:

$$\begin{matrix} \mathbf{R}_3 \leftrightarrow \mathbf{R}_2 \\ \Rightarrow \end{matrix} \begin{bmatrix} 1 & i & -i \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \mathbf{R}_2 / 2 \\ \Rightarrow \end{matrix} \begin{bmatrix} 1 & i & -i \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \mathbf{R}_1 - i\mathbf{R}_2 \\ \Rightarrow \end{matrix} \begin{bmatrix} 1 & 0 & -1.5i \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{e}_0 = \begin{bmatrix} 1.5iz \\ -z/2 \\ z \end{bmatrix}$$

I let you figure out the second eigenspace.

Summary

- All operations and concept that we developed in linear algebra with real numbers can be extended by using complex numbers as scalars.
- Some properties, such as the concept of angle, must be handled with care, while others, like eigenvalues take on new strength.

Common errors to avoid

- Watch out to perform all algebraic operations according to the definition of the imaginary unit i.

Learning questions for Section LA 11-3: Linear algebra with complex numbers

Review questions:

1. Describe how each linear algebra tool and operation is extended to complex numbers. (WARNING: the answer to this question is extremely simple! If your answer is long, you may have missed something!)

Memory questions:

1. Which tools and concepts of linear algebra extend over $\mathbb{C}$?

Computation questions:

Determine the row and column spaces of the following matrices:

1. $\begin{bmatrix} i & i+1 \\ i+2 & i+3 \end{bmatrix}$	2. $\begin{bmatrix} i & 2i \\ 3i & 4i \end{bmatrix}$	3. $\begin{bmatrix} 1-i & 1 \\ 2-i & 2i \end{bmatrix}$	4. $\begin{bmatrix} i & 2 \\ i+1 & 4 \end{bmatrix}$

Determine the inverse of each of the following matrices, or explain why it is not invertible.

5. $\begin{bmatrix} i & i+1 \\ i+2 & i+3 \end{bmatrix}$	6. $\begin{bmatrix} i & 2i \\ 3i & 6i \end{bmatrix}$	7. $\begin{bmatrix} 1-i & i+1 \\ -i & i+2 \end{bmatrix}$	8. $\begin{bmatrix} i & 2 \\ i+1 & 4 \end{bmatrix}$

Compute the determinant of each of the following matrices, or explain why this is not possible to compute it.

9. $\begin{bmatrix} i & i+1 \\ i+2 & i+3 \end{bmatrix}$

10. $\begin{bmatrix} i & 2i \\ 3i & 6i \end{bmatrix}$

11. $\begin{bmatrix} 1-i & 1 \\ 2-i & 2i \end{bmatrix}$

12. $\begin{bmatrix} i & 2 \\ i+1 & 4 \end{bmatrix}$

13. $\begin{bmatrix} i & i+1 \\ 2 & -i \\ 2i & 7 \end{bmatrix}$

14. $\begin{bmatrix} i^2 & \sqrt{i} \\ 2 & -1/i \\ i+1 & e \end{bmatrix}$

Compute the eigenvalues and the corresponding eigenspaces of the following matrices.

15. $\begin{bmatrix} i & 3 \\ 2 & i \end{bmatrix}$

16. $\begin{bmatrix} i & 2i \\ 3i & 6i \end{bmatrix}$

17. $\begin{bmatrix} i & i+1 \\ 2-i & 2i \end{bmatrix}$

18. $\begin{bmatrix} i & 2 \\ i+1 & 4 \end{bmatrix}$

Proof questions:

1. Construct an orthogonal matrix with complex entries and prove that it is orthogonal.

2. Prove that if $\mathbf{A}$ is a matrix with real entries and λ is one of its eigenvalues, then λi is also an eigenvalue of $i\mathbf{A}$ with the same eigenspace.

Templated questions:

In these questions, make your own choice of any items involved.

1. Construct two 4D vectors over the complex numbers and determine their dot product. Use the result to determine if the vectors are orthogonal.

What questions do you have for your instructor?