Chapter 11: Vector spaces — *Section 3*

Vector spaces of matrices

What you need to know already:	What you can learn here:
➤ The ten axioms needed to define a vector space.	➤ How matrices can form vector spaces, both with the usual operations and with unusual ones.

I don't mean to repeat myself, but a central concept to understand in this chapter is that vector spaces were developed so as to generalize properties of Euclidean vectors to other sets. To gently go where other LA students have gone before, we shall now look at vector spaces that consist of objects that are still vectors, but also some added property, namely matrices.

Since matrices are vectors with an added structure, we have good hope that their usual operations can give some of their sets a vector space structure. Of course, in each case we still need to check that the closure axioms hold, both in terms of the operations being well defined and of their keeping their results within the set we are considering.

We shall start with the obvious:

Technical fact

The set of all $m\times n$ matrices, with the usual operations, forms a ***vector space***, denoted by M_{mn}.

Here all we need to observe is that $m\times n$ matrices are really vectors of dimension mn and therefore they are Euclidean vectors in the usual sense, only with the additional row and column structure.

Let us look at something a little more interesting.

Technical fact

The set of $n\times n$ ***diagonal matrices***, with the usual operations, forms a vector space.

Proof

Again, we only need to check the closure axioms, which is easy. Here is the proof without words:

$$\begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & a_n \end{bmatrix} + \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} a_1+b_1 & 0 & \cdots & 0 \\ 0 & a_2+b_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & a_n+b_n \end{bmatrix}$$

$$k\begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} ka_1 & 0 & \cdots & 0 \\ 0 & ka_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & ka_n \end{bmatrix}$$

Just to avoid creating a wrong impression, here is an example of a set of matrices that does NOT form a vector space.

Technical fact

The set of $n \times n$ invertible matrices, with the usual operations, does ***NOT*** form vector space.

Proof

The zero matrix should be the zero vector of this space, but it is not invertible. Therefore, S1 is not true.

These checks look rather easy.

They are, if you remember what you need to check:

- Check all axioms to show that it is a vector space.
- Skip the axioms that are known to be true from previous knowledge about the sets and operations in question.
- Provide one counterexample to show that a structure is NOT a vector space.

But of course, the usual operations are not the only possible ones we can use for the set of matrices. Here is an example of a vector space that uses matrices, but with different operations.

Technical fact

The set of matrices of the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

with operations defined by:

$$\mathbf{A} \oplus \mathbf{B} = \mathbf{AB}$$

$$k\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix}$$

forms a ***vector space***.

Proof

I will check the closure and special items axioms and leave to you to check the others.

To prove that this space is closed under addition, we notice that for any two such matrices:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\sigma & -\sin\sigma \\ \sin\sigma & \cos\sigma \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\theta\cos\sigma - \sin\theta\sin\sigma & -\cos\theta\sin\sigma - \sin\theta\cos\sigma \\ \sin\theta\cos\sigma + \cos\theta\sin\sigma & -\sin\theta\sin\sigma + \cos\theta\cos\sigma \end{bmatrix}$$

By using addition formulae, these turn out to be:

$$\begin{bmatrix} \cos(\theta+\sigma) & -\sin(\theta+\sigma) \\ \sin(\theta+\sigma) & \cos(\theta+\sigma) \end{bmatrix}$$

Therefore, by adding two of these matrices we obtain another matrix of the same form. Notice that you can use this last formula also to check the algebraic axioms.

The scalar multiplication satisfies the closure axiom C2 by definition.

What is the zero vector here? Since we are using matrix multiplication as "*addition*", the identity matrix should work as identity for addition, that is, as the zero vector. By the way, notice the consistency of the terminology for *identity*. The identity matrix is obtained by letting $\theta = 0$, so the zero vector exists and it works as such.

As for the negative of a vector, it is obvious that (a sentence so dear to mathematicians!):

$$-\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

We can see that this is true because:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \oplus \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} =$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \mathbf{I}$$

And this is our zero vector in this vector space.

Finally, the scalar 1 works as the identity for scalar multiplication by definition.

My showing you more examples would only damage your learning, as you need to get your own experience, so, off you go to the *Learning questions*, which will again be only of the *Proof* variety.

Summary

- Since matrices are Euclidean vectors, there are several subsets of them that are vector spaces with the usual operations.
- But there are also sets of matrices that are vector spaces by using different operations.

Common errors to avoid

- Keep watching for those unusual operations. They may seem tricky (nay, nasty!) but they are essential to properly understand the concept of vector spaces.

Learning questions for Section LA 11-3

Proof questions:

For each of the sets of matrices identified in questions 1-8, determine whether the usual operations provide them with a vector space structure.

1. The set of all matrices.
2. The set of all $n \times n$ matrices for a given integer n.
3. The set of $n \times n$ upper triangular matrices.
4. The set of $n \times n$ lower triangular matrices.
5. The set of all $n \times n$ matrices whose trace (sum of the elements on the main diagonal) is 0.
6. The set of all $n \times n$ matrices whose trace (sum of the elements on the main diagonal) is 1.
7. The set of 2×2 matrices of the form $\begin{bmatrix} a & b \\ 3b & -a \end{bmatrix}$.
8. The set of 3×3 matrices of rank 2.
9. Prove that the algebraic axioms hold for the set of matrices of the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with operations defined by $\mathbf{A} \oplus \mathbf{B} = \mathbf{AB}$ and $k\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix}$.
10. Prove that the linear transformation associated with the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ performs a rotation by an angle θ. As a consequence, explain why the set of such rotations forms a vector space and describe its operations.
11. Show that the set of diagonal $n \times n$ matrices satisfies at least **four** of the axioms for a vector space (you can choose which four).

Templated questions:

1. Select a possibly interesting set of matrices and determine whether it is a vector space or not, whether with the usual operations or with some ad hoc ones.

What questions do you have for your instructor?