Chapter 11: Vector spaces **Section 4**

Vector spaces of functions

What you need to know already:	What you can learn here:
➤ Definition and axioms for a vector space. ➤ Basic properties of differentiation and integration.	➤ How certain sets of functions, with the usual operations, also form vector spaces, but without the benefit of the familiar components of a Euclidean space.

It is time to start moving away from the Kansas of usual vectors for good! We shall now see some vector spaces that are really NOT vectors in the sense that has become familiar to you in this course and before. But they will still consist of familiar objects and operations. We start from the most basic one, although it is the biggest one!

Technical fact

The set consisting of ***all*** real valued ***functions***, together with the usual operations of function addition and scalar multiplication, forms a vector space denoted by $\mathfrak{F}$ (that's a fancy "F").

Proof

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For addition, we have:

Closure. The addition of any functions is still a function:

$$(f+g)(x)=f(x)+g(x).$$

Commutativity. The sum of two functions is the same in either order:

$$f(x)+g(x)=g(x)+f(x).$$

Associativity. Since addition of functions follows the rules of the addition of numbers, it is associative:

$$(f+g)(x)+h(x)=f(x)+(g+h)(x)$$

Existence of the zero vector. The constant function $f(x)=0$ acts as the 0 vector in $\mathfrak{F}$.

Existence of all negative vectors. The function $g(x)=-f(x)$ acts as the negative of each vector in $\mathfrak{F}$.

For scalar multiplication, we have:

Closure. The product of any scalar *c* with any function is still a function:

$$(cf)(x) = cf(x)$$

Associativity. This follows again from properties of numbers:

$$(cdf)(x) = c(df)(x) = cdf(x)$$

Neutrality of 1. This is obviously true: $(1f)(x) = 1f(x) = f(x)$

Distributivity over vector addition. This follows from properties of numbers:

$$c(f+g)(x) = cf(x) + cg(x)$$

Distributivity over scalar addition. This also follows easily:

$$(c+d)f(x) = cf(x) + df(x)$$

This sounds too simple: do we have to check those properties that we know are true?

I realize that in this example it was very easy to check the axioms, but it is an important step, since such check will give us the license to use any of the properties of vector spaces that we shall identify in the future.

So, for now, take it as an uninspiring, but necessary warm-up drill and perform an axiom check to prove the following facts.

Technical fact

Each of the following sets of functions, together with the usual operations of function addition and scalar multiplication, forms a vector space:

- The set of all ***continuous*** functions, commonly denoted by $\mathcal{C}$.
- The set of all ***differentiable*** functions, commonly denoted by $\mathcal{D}$.
- The set of ***integrable*** functions.
- The vector space of all ***polynomials***, commonly denoted by $\mathcal{P}$.
- The vector space of all ***polynomials of degree at most n***, commonly denoted by $\mathcal{P}_n$.
- The vector space of all differentiable functions that are ***solutions*** of a differential equation of the form $af''(x) + bf'(x) + cf(x) = 0$.

And you want me to check all these, right?

Yes, but remember that the algebraic and special item axioms always hold true, as they are basic properties of all functions. So, you only need to check the closure axioms and for that you can even just look at the linear combination axiom.

Instead I will show you a special case of the last type of vector spaces.

Example: $2f'' - f' + 3f = 0$

Consider the set of all differentiable functions $y = f(x)$ that are solutions of this differential equation. I claimed in the previous fact box that this set forms a vector space with the usual operations of addition and scalar multiplication. Let's prove it.

Since we are dealing with differentiable functions, the algebraic and special item axioms will certainly work, so we only need to check the closure axioms.

So, let us say that *f* and *g* are two such solutions and let us check if their sum $f+g$ is also a solution:

$$2(f+g)''-(f+g)'+3(f+g)$$
$$=2f''+2g''-f'-g'+3f+3g$$

Now we rearrange the terms:

$$=\left(2f''-f'+3f\right)+\left(2g''-g'+3g\right)$$

Since our functions are both solutions, both brackets equal 0 and hence so does their sum. Therefore, the sum is also a solution.

Similarly, if c is any scalar, then is $cf(x)$ a solution of the equation? Let's see:

$$2(cf)''-(cf)'+3(cf)=2cf''-cf'+3cf=c\left(2f''-f'+3f\right)$$

Once again, the quantity in bracket is 0, which means that $cf(x)$ is a solution.

Since both closure axioms, and hence all axioms, are true, this is a vector space.

Here is a variation on the list I provided in the fact box, just to show you that there are many other possibilities.

Example: $\int_0^1 f(x)dx=0$

The set consisting of continuous functions whose definite integral between 0 and 1 is 0 forms a vector space.

Since this is a subset of the vector space of continuous functions, commutativity, associativity and distributivity work. All we need to check are the closure axioms.

I will use the linear combination axiom. If $f(x)$ and $g(x)$ are functions in this set, we can see that

$$\int_0^1\left(af(x)+bg(x)\right)dx=\int_0^1 af(x)dx+\int_0^1 bg(x)dx$$
$$=a\int_0^1 f(x)dx+b\int_0^1 g(x)dx=a0+b0=0$$

Therefore, this is a vector space.

Example: $\int_0^1 f(x)dx=1$

If we consider the set of continuous functions whose integral between 0 and 1 is 1, the zero vector, that is, the zero function, does not belong there.

Therefore, this set is not a vector space.

So, there are sets of functions that form a vector space and sets that don't. But what does this tell us?

I have no idea!

Well, let's see if *this* opens an interesting new avenue: we saw earlier that all Euclidean spaces and their subspaces have bases, each consisting of finitely many vectors. Do you remember what we call the number of vectors in such a basis?

The dimension of the subspace.

Right. So, what can we use as a basis for, say. the vector space $\mathcal{P}$ of polynomials?

Do you mean all polynomials?

Yes.

Well, I guess we need all possible powers of the variable, since linear combinations do not change the degree.

Right, so how many of those powers do we need?

Oh, infinitely many!

Right, so the concept of dimension needs to be revisited.

Technical fact

A vector space may not have a finite set of vectors that span it, so that the concepts of ***dimension*** and ***basis***, together with the others that are linked to them, need to be revised in a generic vector space.

Does that mean also that there are other features that we have studied so far in relation to vectors, but need to be revised for general vector spaces?

Definitely! And as you can imagine, that is a very big task. I will outline some of its parts in later sections, but you will probably have to wait for later courses to see them more in depth. It involves many major issues that have led mathematicians to many interesting and surprising discoveries.

Also, deciding closure is not always easy, since we do not have matrices to rescue us through some standard process. Here is an example.

Example: $y = a\sin(bx)$

Is the set of all function of this form, with *a* and *b* being any real numbers, a vector space? Notice that the 0 function is here, so we cannot take this easy way out. Something tells us that this does not smell like a vector space, but how do we prove it?

We can try to construct a linear combination of two such functions and show that it is not of the same form. That's what we'll do, but it is not as easy as it sounds. We try with the linear combination of $\sin(x)$ and $\sin(2x)$ (keep it simple!) given by their sum. Can we write it as a member of our set? Let's say that:

$$\sin(x)+\sin(2x) = c\sin(dx)$$

If this were true, we would have, for $x=\pi$:

$$\sin(\pi)+\sin(2\pi) = c\sin(d\pi) = 0$$

But $c \neq 0$, since the sum of our functions is not the zero function, so that we need *d* to be a whole number. But then, for $x = \frac{\pi}{2}$ we would need:

$$\sin\left(\frac{\pi}{2}\right)+\sin(\pi) = c\sin\left(d\frac{\pi}{2}\right) = 1$$

If *d* is even, this is impossible, as it would give us 0=1! If it is odd, say $d = 2k+1$, we would need $c = \pm 1$. But in that case:

$$\sin(x)+\sin(2x) = \pm\sin((2k+1)x)$$

But the maximum value for the function on the right side is 1, while for the left it can be bigger than 1 (check that on the calculator at least!)

Therefore, this linear combination cannot be written as a function in the set and closure fails.

Wow! A lot of work just to prove one case.

And fortunately one counterexample is all we need. Just imagine how messy things can get.

But before jumping too far, it is time for you to get to better know some basic vector spaces of functions, as well as to find out which ones are not worthy of the name.

Summary

- By using the common operation of addition and scalar product, several sets of functions form a vector space.
- Unlike Euclidean spaces, some of these vector spaces need infinitely many vectors to be spanned completely. This concept needs deeper and more careful analysis.

Common errors to avoid

- Not every set of functions forms a vector space: always check the axioms, or show that one does not work. Don't jump to conclusions based on feelings!

Learning questions for Section LA 11-4

Proof questions:

For each of the sets of functions presented in questions 1-20 determine if the set, with the usual operations, forms a vector space.

1. The set of all polynomials of degree n.
2. The set of rational functions.
3. The set of exponential functions of the form $y = a^x$, with $a \geq 0$.
4. The set of functions of the form $y = k\, a^x$, where *a* is a fixed positive number and *k* can be any real number.
5. The set of hyperbolae of the form $y = \frac{a}{x} + b$, a and b being any real numbers.
6. The set of logarithmic functions of the form $y = \ln x^a$.
7. The set of all hyperbolic functions of the form $y = \cosh(kx)$.
8. The set of all hyperbolic functions of the form $y = \sinh(kx)$.
9. The set of trigonometric functions of the form $y = \sin(ax)$, with *a* any real number.
10. The set of functions of the form $y = a \sin x$, with *a* any real number.
11. The set $\mathcal{C}$ of continuous function.
12. The set of functions continuous on $[0,1]$.
13. The set $\mathcal{D}$ of differentiable functions.
14. The set $\mathcal{P}$ of polynomial functions.
15. The set $\mathcal{P}_n$ of polynomial functions of degree at the most *n*.
16. The set of all second degree polynomials whose coefficients add up to 0.

17. The set of all second degree polynomials whose coefficients add up to 1.

18. The set of differentiable functions that are solutions of a differential equation of the form $af''(x)+bf'(x)+cf(x)=0$, where a, b and c are scalars.

19. The set of polynomials containing three given points.

20. The set of polynomials containing three given x-intercepts.

21. The subset of $\mathcal{P}_4$ consisting of those polynomials for which the sum of the coefficients of the even powers is 0.

22. The subset of $\mathcal{P}_4$ consisting of those polynomials for which the sum of the coefficients of the even powers is 1.

What questions do you have for your instructor?