Chapter 11: Vector spaces **Section 5**

The vector space of Complex numbers

What you need to know already:	What you can learn here:
➤ All basic algebra of real numbers. ➤ Vector space axioms.	➤ What imaginary and complex numbers are. ➤ Their basic algebraic properties.

You have most likely heard of imaginary and complex numbers in the past and most introductory courses on linear algebra include them, so why not start exploring this mysterious topic now, in the context of vector spaces?

Because I have a date? ☺

Poor excuse! Complex numbers will show up repeatedly in your engineering training, in calculus as well as in other areas, such as physics, so learning about them is very useful.

But why in the context of vector spaces?

Because they provide another nice example of a vector space not consisting of usual vectors. Moreover, they are fun, really! Their properties take off from familiar territory and move on to the unfamiliar, the unexpected and the truly puzzling.

But for now, once again, we'll only be able to open the door behind which is such a huge and exciting mathematical land. I sincerely hope that you will learn more about complex numbers in the future, as they are truly magnificent!

You sound very excited, so what are these complex numbers?

It all started from what seemed like an insurmountable barrier: there is no real number whose square is negative. Having assumed for many centuries that considering numbers with a negative square was reserved for psychiatric cases, mathematicians finally had a brilliant idea: let us *invent* a number whose square is -1 and, to be on the sane side of life, let us call it an imaginary unit.

Definition

The symbol ***i*** denotes the ***imaginary unit***, that is, an entity whose square is -1:

$$i^2 = -1$$

If b is a real number, any multiple of i, that is, any object of the form ***bi*** is called an ***imaginary number***.

But imagining such a number does not force it to exist!

Careful: there is no such *real* number and we don't claim that this *thing* called i is a real number. Why, that is the reason for calling it imaginary! So, we don't have to prove that such a number exists: we *define* i and elevate it to the status of number, albeit imaginary.

But what's the point of imagining things?

We are not imagining them, but defining them. When the idea was first proposed, mathematicians scoffed at it as well, claiming that it was a stupid waste of time. But then something happened: by using imaginary numbers, many properties of real numbers that were mysterious before could now be proved and, with them, several physics phenomena also became clearer. So, imaginary numbers may not show up in our everyday functioning, but they prove to be a great tool in the study of mathematics and science in general.

To be more accurate, what proves to be really useful is the following concept.

Definition

If a and b are real numbers, an entity of the form $a+bi$ is called a ***complex number***.

A complex number is often denoted by the generic letter ***z***, as in $z=a+bi$

The numbers a and b are called, respectively, the ***real*** and ***imaginary parts*** of $z=a+bi$ and are usually denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ respectively.

Example:

Well, not much to show you yet, but, just for the record, the following are real numbers:

$$2, -3, \frac{1}{4}, 5\pi, \sqrt{6}, e^7$$

The following are imaginary numbers:

$$2i,\ -3i,\ \frac{1}{4}i,\ 5\pi i,\ \sqrt{6}i,\ e^7 i$$

The following are complex numbers:

$$2-3i,\quad \frac{1}{4}+5\pi i,\quad \sqrt{6}+e^7 i$$

And, for the second of them:

$$\mathrm{Re}\left(\frac{1}{4}+5\pi i\right)=\frac{1}{4}\quad ;\quad \mathrm{Im}\left(\sqrt{6}+e^7 i\right)=e^7$$

Since we defined i as a number, albeit an imaginary one, and since we defined complex numbers as linear combinations of real and imaginary numbers, we can extend our definitions to include basic arithmetic operations, which end up working well just by mimicking the usual ones.

Definition

The sum of two imaginary numbers is obtained by factoring i and adding the corresponding scalars:

$$ci+di=(c+d)i\quad ;\quad ci-di=(c-d)i$$

The sum of two complex numbers is obtained by collecting like terms:

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

The product of two complex numbers is obtained by using common expansion rules and remembering the $i^2=-1$:

$$(a+bi)(c+di)=ac+adi+bci+bdi^2$$
$$=(ac-bd)+(ad+bc)i$$

The product of a real and an imaginary or complex number are defined, respectively and naturally, by setting:

$$a(bi)=(ab)i\quad ;\quad a(c+di)=ac+adi$$

The power of a complex number is obtained through the natural extension of products to them:

$$(a+bi)^2 = a^2 + 2abi + b^2 i^2 = (a^2 - b^2) + 2abi$$

Example:

The following are basic examples of operations with complex numbers:

$$(2-3i)+(5+6i)=7+3i$$

$$(2-3i)(5+6i)=10+12i-15i-18i^2=28-3i$$

$$(2-3i)^2=4-12i+9i^2=-5-12i$$

We are now ready for the first mild surprise about the imaginary unit: once we have defined it, we discover that there are other numbers whose square is -1.

Technical fact

The square of any odd power of i also equals -1:

Proof

We just need to check the claim:

$$\left(i^{2n+1}\right)^2 = i^{4n+2} = i^{4n} i^2 = \left[\left(i^2\right)^2\right]^n (-1) = \left[(-1)^2\right]^n (-1) = -1$$

That is similar to saying that any odd power of -1 also has a square equal to 1. But any such power is still equal to -1. So, does that mean that all odd powers of i equal i?

No! Notice that $i^3 = i^2 i = -i$, but $i^5 = i^4 i = (-1)^2 i = i$. So, I leave to you to figure out the pattern and to convince yourself that i and $-i$ are not the same quantity. Following your analogy, notice that the square of 1 is also 1, but that -1 is not equal to 1!

In the same way, notice that once we define an irrational number whose square is 2 to be $\sqrt{2}$ we find that there is another one, namely $-\sqrt{2}$, with the same property. However, the consequences for complex numbers are even more extensive.

How does linear algebra come in?

Technical fact

The set of ***imaginary numbers*** with the operations defined here forms a ***vector space*** equivalent to $\mathbb{R}$.

The set of ***complex numbers*** with the operations defined here forms a ***vector space*** equivalent to $\mathbb{R}^2$ and denoted by $\mathbb{C}$.

Proof

In a vector space, we only need the operations of addition and scalar multiplication, and we have defined them just as in regular algebra, so it can be easily checked that all axioms are satisfied. (You may want to do it, for good practice!)

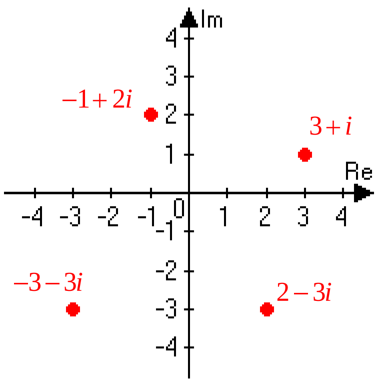
Moreover, with respect to the chosen operations, the set of imaginary numbers is just the set of reals to which a symbol i has been attached.

As for the complex numbers, the real and imaginary parts can be seen as the two components of a 2D vector, with all operations working exactly the same way:

$$a+bi \quad \leftrightarrow \quad \begin{bmatrix} a & b \end{bmatrix}$$

Notice that the real (pun intended) difference that arises with complex numbers is in the product and powers, but these are not part of the vector space structure.

This equivalency between imaginary numbers and real numbers, as well as between $\mathbb{C}$ and $\mathbb{R}^2$, has led to the traditional representation of complex numbers as points in $\mathbb{R}^2$, with the real part corresponding to the horizontal axis and the imaginary part corresponding to the vertical axis.



But there are many other connections that would take years to explore. Again, we are just scratching the surface.

A very basic and intriguing connection exists between the product of complex numbers, which is not involved in the vectors space structure, and determinants, which are also not involved in the vector space structure of Euclidean spaces, but sneak in in other ways.

Technical fact

The product of two complex numbers can be obtained as:

$$(a+ib)(c+id) = \begin{vmatrix} a & b \\ d & c \end{vmatrix} + i \begin{vmatrix} a & b \\ -c & d \end{vmatrix}$$

Is this also related to the dot product?

Does it look like one? The formula is rather different and more complex. Moreover, dot products are not part of a vector space structure, as they are not a consequence of the axioms, so we cannot expect them to show up all the times. We shall see later how this can be done in general, but not here, not yet.

Let's look at some other connections.

Definition

The ***conjugate*** of a complex number $z = a+ib$ is the complex number $\overline{z} = a - ib$.

Example:

Here are the conjugates of some good old friends of ours:

$$z = 2-3i \quad \Rightarrow \quad \overline{z} = 2+3i$$

$$z = \frac{1}{4} + 5\pi i \quad \Rightarrow \quad \overline{z} = \frac{1}{4} - 5\pi i$$

$$z = \sqrt{6} + e^7 i \quad \Rightarrow \quad \overline{z} = \sqrt{6} - e^7 i$$

This looks like the conjugates for radicals!

It is exactly the same and it can be used just as effectively. For instance, we can use conjugates to recover the concept of magnitude of a complex number, similar to that of a Euclidean vector. But since this concept was studies within the context of complex number, mathematicians gave it a yet different name, rejecting length, magnitude and norm!

Definition

The ***modulus*** of a complex number $z = a + bi$ is its norm when viewed as a 2D vector, but it is denoted as an absolute value:

$$|z| = \sqrt{a^2 + b^2}$$

And with this definition, the following fact is rather obvious.

Technical fact

The ***product*** of a complex ***number*** with its ***conjugate*** provides the square of the modulus of z:

$$z\,\overline{z} = (a + ib)(a - ib) = a^2 + b^2$$

Example:

Here are the *moduli* (plural of modulus!) of our good old friends:

$$|2 - 3i| = \sqrt{4+9} = \sqrt{13}$$

$$\left|\frac{1}{4} + 5\pi i\right| = \sqrt{\frac{1}{16} + 25\pi^2} \quad ; \quad \left|\sqrt{6} + e^7 i\right| = \sqrt{6 + e^{14}}$$

Notice that we have not yet defined a way of dividing by an imaginary or complex number, since i is really just a strange object that we are combining with real numbers. But it turns out that there is a way to implement a division by complex numbers if we use conjugates and moduli. We can do that by, once again, assuming that what we are looking for exists and by exploring its properties.

So, if it were possible to compute a ratio of the form $\frac{z_1}{z_2}$, with $z_2 \neq 0$, it would imply that:

$$\frac{z_1}{z_2} = \frac{z_1\overline{z}_2}{z_2\overline{z}_2} = \frac{1}{|z_2|^2} z_1\overline{z}_2$$

But the expression on the right makes perfect sense, since it is obtained as the product of two complex numbers, a perfectly fine operation, combined with multiplication by a real scalar, also a perfectly good operation. So, we have a way!

Definition

Given two complex numbers z_1 and $z_2 \neq 0$, we define their ***quotient*** as:

$$\frac{z_1}{z_2} = \frac{z_1\overline{z}_2}{|z_2|^2}.$$

Notice that what we are doing when dividing two complex numbers is, in a way, rationalizing the denominator!

Example: $\dfrac{2-3i}{5+i}$

By using our definition, we get:

$$\frac{2-3i}{5+i}=\frac{(2-3i)(5-i)}{(5+i)(5-i)}=\frac{7-17i}{26}=\frac{7}{26}-\frac{17}{26}i$$

But is this what we want? That is, if we multiply this quotient by the denominator, do we get the numerator? Try that, it is a good exercise!

Example: $\dfrac{\sqrt{6}+i}{2-\sqrt{3}i}$

Same idea and method, just slightly more messy computations.

$$\frac{\sqrt{6}+i}{2-\sqrt{3}i}=\frac{(\sqrt{6}+i)(2+\sqrt{3}i)}{(2-\sqrt{3}i)(2+\sqrt{3}i)}=$$

$$=\frac{2\sqrt{6}-\sqrt{3}+(2+3\sqrt{2})i}{7}=\frac{2\sqrt{6}-\sqrt{3}}{7}+\frac{2+3\sqrt{2}}{7}i$$

The fact that complex numbers form a 2-dimensional vector space is used extensively in calculus. In fact, calculus of complex variables is an extremely fascinating area of mathematics which leads to many important applications, despite its origin in imaginary numbers.

Moreover, complex numbers are successfully used as scalars in an even more extended version of the concept of vector spaces!

And watch out for the end of your second term calculus course: you will discover a relationship that is commonly considered as the most beautiful equation in mathematics, namely:

$$e^{i\pi}+1=0$$

What is it the most beautiful and why is it true?

Oh, I would love to talk about this and much more about complex numbers, but then you would curse at the time of the final exam!

So, let's stop here for now. You will see complex numbers again as you explore more calculus and linear algebra concepts and I hope I have left you with enough interest to continue studying them. They may be called complex and imaginary, but they are very concrete and fun to work with.

Summary

- Imaginary and complex numbers started from the desire to have a square root for -1. They have led to previously unimaginable discoveries in math and science.
- The definition is simple, the operations as familiar, the consequences are enormous!

Common errors to avoid

- Don't get bogged down by the initial strangeness of the concept. Accept it, work with it and learn to enjoy the whole!

Learning questions for Section LA 11-5

Review questions:

1. Explain what an imaginary number is.
2. Describe what a complex number is.
3. Compare and contrast the concepts of imaginary and complex numbers.
4. Describe how to perform addition, subtraction, multiplication and division between real, imaginary and complex numbers.
5. Explain how complex numbers form a vector space.

Memory questions:

1. What property identifies the imaginary unit *i*?
2. What formula identifies a complex number?
3. Which operations are used to endow $\mathbb{C}$ with a vector space structure?
4. How is the quotient of two complex numbers defined?

Computation questions:

Given the introductory nature of this section, the only questions involving computations are of a simple nature and are pointed at in the *Templated questions* section.
Here are two questions that link to a basic concept not mentioned in the body of the section.

1. Determine the reciprocal of the complex number $3+4i$
2. Determine the reciprocal of the complex number $4-3i$.

Theory questions:

1. Does the definition of quotient of two complex numbers work as it does for real numbers when the denominator is a real number?

Proof questions:

1. Prove that for any two complex numbers z_1, z_2 it is true that $|z_1 z_2| = |z_1||z_2|$.

2. Show that a formula for factoring a sum of squares can when allowing complex numbers.

3. By using the correspondence between the set $\mathbb{C}$ of complex numbers and $\mathbb{R}^2$, show how a complex number can be written in terms of polar coordinates.

4. Following on the previous question, show that every complex number $a+bi$ can be written as $\sqrt{a^2+b^2}\left(\cos\theta + i\sin\theta\right)$ where $\theta = \tan^{-1}\frac{b}{a}$. This is called the polar form of a complex number.

5. Use the claim of the previous question and the Weierstrass substitution to prove that every complex number of modulus 1, with one exception only, can be written as $\frac{1+it}{1-it}$, where *t* is a real number. Which number is the exception?

6. Show that the linear transformation $T:\mathbb{C}\to\mathbb{C}$ that changes each complex number $a+bi$ to $b+ai$ is linear by showing that it preserves linear combinations.

7. Show that the linear transformation $T:\mathbb{C}\to\mathbb{C}$ that changes each complex number to its conjugate is linear by showing that it preserves linear combinations.

8. Show that the linear transformation $T:\mathbb{C}\to\mathbb{C}$ that changes each complex number to its product with a fixed complex number *z* is linear by showing that it preserves linear combinations.

9. Show that the linear transformation $T:\mathbb{C}\to\mathbb{C}$ that changes each complex number to its division by a fixed, non-zero complex number *z* is linear by showing that it preserves linear combinations.

10. The set of upper triangular 3×3 matrices with entries consisting of complex numbers forms a vector space. Pick any four of the axioms of a vector space and show that they apply to these matrices as vectors.

Templated questions:

1. For any two complex numbers z_1, z_2 of your choice:
   a) Plot the numbers in the Cartesian plane.
   b) Compute their sum, difference and product.
   c) Compute their modulus and conjugate.
   d) Verify that the product of their moduli is the modulus of their product.
   e) Compute their quotient both ways.
   f) Determine the polar form of each of them (see Proof questions 3 and 4)

2. Pick a complex number *z* and, by using the correspondence between $\mathbb{C}$ and $\mathbb{R}^2$, determine the matrices that represent:
   a) multiplication by z
   b) division by z

What questions do you have for your instructor?