Chapter 2: Euclidean vectors **Section 2**

Algebraic operation with Euclidean vectors

What you need to know already:	What you can learn here:
➤ The algebraic operations that can be done with geometric vectors.	➤ How to extend such operations to all Euclidean vectors.

Since Euclidean vectors are just longer than geometric vectors, but with the same overall look, it is very a simple task to apply to them the operations we have constructed for geometric vectors.

Definition

The ***sum*** of two *n*-dimensional Euclidean vectors is the vector obtained by adding the corresponding components of the vectors:

$$\mathbf{v}+\mathbf{w}=\begin{bmatrix}v_1 & v_2 & \dots & v_n\end{bmatrix}+\begin{bmatrix}w_1 & w_2 & \dots & w_n\end{bmatrix}$$

$$=\begin{bmatrix}v_1+w_1 & v_2+w_2 & \dots & v_n+w_n\end{bmatrix}$$

The same principle applies to the definition of the sum of ***several*** Euclidean vectors of the same dimension.

Example:

$$\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}+\begin{bmatrix}-2 & 3 & -1 & 2\end{bmatrix}+\begin{bmatrix}0 & 1 & -1 & 2\end{bmatrix}=\begin{bmatrix}-1 & 6 & 1 & 8\end{bmatrix}$$

The definition states that this addition can only be done among vectors of the same dimension? Is it possible to generalize that?

I am glad to see that you are quickly catching on the idea of generalization! In most cases we shall need to add only vectors of the same dimension, but occasionally it is useful to play with vectors of different dimensions. In that case we shall use this simple algebraic procedure.

Definitions

An *n*-dimensional vector **v** is ***embedded*** in $\mathbb{R}^{n+p}$ by adding *p* components all equal to 0. Unless otherwise needed, the additional components are added at the end.

A vector **v** in $\mathbb{R}^{n+p}$ can be ***truncated*** to an *n*-dimensional vector by deleting the *p* components that are not wanted or needed. When the deleted components are the last ones, this operation is also called a ***projection*** of $\mathbb{R}^{n+p}$ on $\mathbb{R}^n$.

Example:

We can embed the vector $\mathbf{v}=\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}$ into $\mathbb{R}^6$ by adding two zeros at the end and identifying it with the 6-dimensional vector $\mathbf{v}'=\begin{bmatrix}1 & 2 & 3 & 4 & 0 & 0\end{bmatrix}$.

Similarly, we can truncate **v** to a 3 dimensional vector by dropping the second component and identifying it with the 3-dimensional vector $\mathbf{v}''=\begin{bmatrix}1 & 3 & 4\end{bmatrix}$.

Or, we can project it on $\mathbb{R}^3$ and identify it with the vector $\mathbf{v}'''=\begin{bmatrix}1 & 2 & 3\end{bmatrix}$.

How about scalar multiplication? Can't it be generalized too?

Of course!

Definition

The ***scalar multiplication*** of a scalar *c* and a Euclidean vector **v** is obtained by multiplying each of the components of **v** by *c*:

$$c\mathbf{v}=c\begin{bmatrix}v_1 & v_2 & \dots & v_n\end{bmatrix}=\begin{bmatrix}cv_1 & cv_2 & \dots & cv_n\end{bmatrix}$$

Example:

$$5\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}=\begin{bmatrix}5 & 10 & 15 & 20\end{bmatrix}$$

Definition

Two vectors **u** and **v** in the same $\mathbb{R}^n$ are ***scalar multiples*** of each other if there is a scalar *c* such that *c***u**=**v** or *c***v**=**u**.

Example:

The vectors $\mathbf{v}=\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}$ and $\mathbf{u}=\begin{bmatrix}2 & 4 & 6 & 8\end{bmatrix}$ are scalar multiples of each other, since $\mathbf{u}=2\mathbf{v}$.

The vectors $\mathbf{v}=\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}$ and $\mathbf{w}=\begin{bmatrix}-3 & -6 & -9 & -12\end{bmatrix}$ are also scalar multiples since $\mathbf{w}=-3\mathbf{v}$.

However, the vectors $\mathbf{v}=\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}$ and $\mathbf{z}=\begin{bmatrix}2 & 3 & 4 & 5\end{bmatrix}$ are not parallel, since the ratio of the two first components is not the same as the ratio of the second components (or any other pair, for that matter), so they cannot be multiples of each other.

But in this way the zero vector is a scalar multiple of any vector, since $\mathbf{0}=0\mathbf{v}$ *!*

Yes, and that is not a bad thing, as we shall see later. Just for the record:

Definitions

The vector $\mathbf{0}_n=\begin{bmatrix}0 & 0 & \dots & 0\end{bmatrix}$ in $\mathbb{R}^n$ is called the ***zero vector*** and any other vector is said to be a ***non-zero vector***.

Technical fact

The vector $\mathbf{0}_n$ is a scalar multiple of any other vector in $\mathbb{R}^n$.

The next definition is also an easy extension, but I am isolating it because it refers to the most important concept of linear algebra, so it deserves a place of honor:

Definition

A Euclidean vector $\mathbf{w}$ in $\mathbb{R}^n$ is a ***linear combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ of $\mathbb{R}^n$ if there are scalars $c_1, c_2, \ldots, c_k$ such that:

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$$

Example:

The vector $\mathbf{v} = \begin{bmatrix} 5 & 6 & 7 & 8 \end{bmatrix}$ is a linear combination of $\mathbf{u} = \begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 6 & 4 & 10 & 8 \end{bmatrix}$, since:

$$\begin{bmatrix} 5 & 6 & 7 & 8 \end{bmatrix} = 2\begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 6 & 4 & 10 & 8 \end{bmatrix}$$

How did you figure out that 2 and ½ were the needed scalars?

In truth, I cheated, since I constructed the vectors so that the relationship would hold. But later on you shall learn a sure-fire method to check if a vector is a linear combination of other vectors and, if so, which linear combination it is.

Be careful to distinguish the number of vectors involved in the linear combination (k in the definition) from the number of components involved in the Euclidean space being used (n). They need not be the same and therefore we absolutely must use different letters for them.

For instance, in the last example, $k = 2$, but $n = 4$.

What about dot products?

They also seem easily extendable, don't they? And they are:

Definition

The ***scalar product***, or ***dot product*** of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ is denoted by $\mathbf{u} \bullet \mathbf{v}$ and is computed as the sum of the products of the components in corresponding positions. In formula:

$$\mathbf{u} \bullet \mathbf{v} = \begin{bmatrix} u_1 & u_2 & \ldots & u_n \end{bmatrix} \bullet \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$$
$$= u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

Notice once again that the dot product of two vectors is a scalar.

And, of course, the dot product in any dimension has the same properties it has for geometric vectors. Here is a reminder of such properties.

Technical fact

The dot product in $\mathbb{R}^n$ is ***commutative***, ***distributive***, and ***positive definite***.

Also, it is ***associative with respect to scalar multiplication*** in the sense that:

$$c(\mathbf{u}\bullet\mathbf{v}) = (c\mathbf{u})\bullet\mathbf{v} = \mathbf{u}\bullet(c\mathbf{v})$$

However, it is ***NOT associative***, since $(\mathbf{u}\bullet\mathbf{v})\bullet\mathbf{w}$ is not well defined.

I am afraid that examples on this last definition and fact may insult your intelligence, so I refer you to the exercises to help you ensure that the concepts and skills involved are clear to you.

Nothing much new indeed!

Very little: aren't these generalizations easy?

Dare I say almost boring? Yaaaawn!

Before you fall asleep, let me refer you to the next section: it will offer more generalization that are still easy to implement, but lead to some surprising discoveries.

Summary

- All algebraic operations defined for geometric vectors extend in the obvious way to Euclidean vectors.
- Euclidean vectors can be embedded into vectors of higher dimension or truncated into vectors or smaller dimension.

Common errors to avoid

- Don't lower your guard just because these operations look easy: they are, but all the more reason to do them well!

Learning questions for Section LA 2-1

Review questions:

1. Explain how to extend the algebraic operations of geometric vectors to Euclidean vectors.

Memory questions:

1. What is the technical name of an expression of the type $3\mathbf{u}-\mathbf{v}+2\mathbf{w}$?

Computation questions:

1. Given the vectors $\mathbf{u}=\begin{bmatrix}-3 & 1 & 2 & 4\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}4 & 0 & -8 & 1\end{bmatrix}$ and $\mathbf{w}=\begin{bmatrix}6 & -1 & -4 & 3\end{bmatrix}$, compute:
   a) $3\mathbf{u}-\mathbf{v}+2\mathbf{w}$
   b) $\mathbf{u}\bullet\mathbf{w}$
   c) a vector perpendicular to $\mathbf{w}$.

2. Given the vectors $\mathbf{u}=\begin{bmatrix}4 & -6 & 10 & 2\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}18 & 6 & -9 & 3\end{bmatrix}$, $\mathbf{w}=\begin{bmatrix}-25 & 5 & 15 & -10\end{bmatrix}$ determine the vector $\mathbf{z}=\frac{1}{2}\mathbf{u}-\frac{1}{3}\mathbf{v}+\frac{1}{5}\mathbf{w}$

3. Given the vectors $\mathbf{u}=\begin{bmatrix}4 & -6 & 10 & 2\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}18 & 6 & -9 & 3\end{bmatrix}$, $\mathbf{w}=\begin{bmatrix}-25 & 5 & 15 & -10\end{bmatrix}$ determine the value of $\left(\frac{1}{2}\mathbf{u}-\frac{1}{3}\mathbf{v}\right)\bullet\frac{1}{5}\mathbf{w}$ or explain why it does not exist.

Theory questions:

1. When is a dot product of two vectors not possible?

2. For what vectors does $\mathbf{u}\bullet\mathbf{v}+\mathbf{w}$ exist?

3. Is the dot product of Euclidean vectors commutative?

Proof questions:

1. Prove that addition of Euclidean vectors is commutative and associative.

2. Determine whether in an n-dimensional Euclidean space scalar multiplication is distributive with respect to the dot product.

3. Determine whether in an n-dimensional Euclidean space dot product is associative.

4. Prove that scalar multiplication is distributive with respect to vector addition. In particular, make sure you are clear on what I am asking!

5. Prove that for any scalar k and any n-dimensional vectors **u**, **v**, **w**:

$$k(\mathbf{u}+\mathbf{v})\bullet\mathbf{w}=\mathbf{u}\bullet(k\mathbf{w})+(k\mathbf{v})\bullet\mathbf{w}$$

   You can assume basic properties of real numbers, but cannot assume the same properties for vectors: you have to prove them.

Templated questions:

1. Pick a dimension n and two vectors in $\mathbb{R}^n$ and compute their sum.

2. Pick a dimension n, a scalar k and a vector **v** in $\mathbb{R}^n$ and compute $k\mathbf{v}$.

3. Pick a dimension n, k vectors in $\mathbb{R}^n$ and k scalars and compute the corresponding linear combination.

4. Pick a dimension n and two vectors in $\mathbb{R}^n$ and compute their dot product.

What questions do you have for your instructor?