Chapter 2: Euclidean vectors *Section 3*

Geometric operations with Euclidean vectors

What you need to know already:	What you can learn here:
➤ How perform algebraic operations on Euclidean vectors. ➤ How to perform geometric operations on geometric vectors.	➤ How to extend geometric operations to Euclidean vectors.

Higher dimensional vectors are not geometric, therefore it may seem a waste of time to look at their geometric properties. But once again, just because we cannot visualize these vectors or their properties, it does not mean that such properties do not exist or are useless. So, let us attempt to identify these properties and wait patiently to see where they lead.

The definitions are still very simple generalization.

Definition

Two vectors **u** and **v** of $\mathbb{R}^n$ are said to have the same ***direction***, or to be ***parallel***, if and only if they are scalar multiples of each other.

Two parallel vectors have the same ***orientation*** if and only if the scalar factor that relates them is positive.

The vector $\mathbf{0}_n$ is assumed to be parallel to any other vector in $\mathbb{R}^n$.

Examples:

The vectors $\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 2 & 4 & 6 & 8 \end{bmatrix}$ are parallel and have the same orientation, since $\mathbf{u} = 2\mathbf{v}$.

The vectors $\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -3 & -6 & -9 & -12 \end{bmatrix}$ are parallel and have opposite orientations, since $\mathbf{w} = -3\mathbf{v}$.

The vectors $\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} 2 & 3 & 4 & 5 \end{bmatrix}$ are not parallel, since the ratio of the first components is not the same as the ratio of the second components (or any other pair, for that matter), so they cannot be parallel.

Notice that this time the concepts of direction and parallelism make some kind of sense, but are abstract, since we do not have a visual representation of these vectors. The same applies to the following concepts.

Definitions

The ***magnitude*** of a Euclidean vector $\mathbf{v} = [v_1 \ \ v_2 \ \ \ldots \ \ v_n]$ is given by:

$$\|\mathbf{v}\| = \|[v_1 \ \ v_2 \ \ \ldots \ \ v_n]\| =$$

$$= \sqrt{\mathbf{v} \bullet \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

A Euclidean ***unit vector*** is a vector whose magnitude is 1.

A Euclidean unit vector whose components are 1 in position i and 0 everywhere else is called a ***standard*** unit vector and is denoted by the symbol $\mathbf{e}_i$.

Example:

The magnitude of the vector $\mathbf{v} = [1 \ \ -2 \ \ 2 \ \ 3]$ is:

$$\|\mathbf{v}\| = \|[1 \ \ -2 \ \ 2 \ \ 3]\| = \sqrt{1^2 + (-2)^2 + 2^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

The vector $\mathbf{w} = \begin{bmatrix} \frac{-1}{3\sqrt{2}} & \frac{2}{3\sqrt{2}} & \frac{-2}{3\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ is a unit vector, since

$$\|\mathbf{w}\| = \sqrt{\left(\frac{-1}{3\sqrt{2}}\right)^2 + \left(\frac{2}{3\sqrt{2}}\right)^2 + \left(\frac{-2}{3\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{1}{18} + \frac{4}{18} + \frac{4}{18} + \frac{9}{18}} = 1$$

Example:

In $\mathbb{R}^5$ the standard unit vectors are:

$$\mathbf{e}_1 = [1 \ \ 0 \ \ 0 \ \ 0 \ \ 0], \ \mathbf{e}_2 = [0 \ \ 1 \ \ 0 \ \ 0 \ \ 0], \ \mathbf{e}_3 = [0 \ \ 0 \ \ 1 \ \ 0 \ \ 0]$$

$$\mathbf{e}_4 = [0 \ \ 0 \ \ 0 \ \ 1 \ \ 0], \ \mathbf{e}_5 = [0 \ \ 0 \ \ 0 \ \ 0 \ \ 1]$$

Why are we calling it "magnitude" instead of length?

To emphasize the fact that these are not geometric vectors, so this quantity represents – in some way – the size of the vector, but not its traditional length. Having said this, many people do use the word *length* for higher dimensional vectors, so I may also use it. However, your competence in matters vectorial will be better displayed if you use the more specific term *magnitude* ☺. In fact, there is another word that is used in linear algebra circles to denote the same quantity.

Definitions

The magnitude of a Euclidean vector is also called the ***norm*** of the vector.

More items for my dictionary! But does it have the same properties?

Technical facts

For ***any*** Euclidean vector **v** and ***any*** scalar k:

$$\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$$

The magnitude of a Euclidean vector is always ***positive***.

For ***any non-zero*** Euclidean vector **v**, the vector $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a ***unit*** vector.

The proofs of *Facts* 2.3.8 are very simple and I leave them for you in the exercises.

Example:

Since the magnitude of the vector $\mathbf{v} = \begin{bmatrix} 1 & -2 & 4 & 3 \end{bmatrix}$ is:

$$\|\mathbf{v}\| = \left\| \begin{bmatrix} 1 & -2 & 4 & 3 \end{bmatrix} \right\| = \sqrt{1^2 + (-2)^2 + 4^2 + 3^2} = \sqrt{30}$$

the magnitude of the vector $\mathbf{w} = \begin{bmatrix} -5 & 10 & -20 & -15 \end{bmatrix}$, which is equal to -5**v**, is equal to $-5\sqrt{30}$, as you may want to check directly. Moreover, its unit vector is:

$$\mathbf{e}_{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{30}}\begin{bmatrix} 1 & -2 & 4 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{30}} & -\frac{2}{\sqrt{30}} & \frac{4}{\sqrt{30}} & \frac{3}{\sqrt{30}} \end{bmatrix}$$

You may also want to check that this is indeed a unit vector.

You promised something new in this section: where is it?

One surprise coming up! Even though we are dealing with ordered sets of numbers without an obvious geometrical interpretation, we can combine the generalizations seen so far to define an angle between two Euclidean vectors! That is, we can create geometry out of objects that are not naturally inclined to be geometric.

Definition

The ***angle*** θ between two vectors **u** and **v** in $\mathbb{R}^n$ is given by the formula:

$$\theta = \cos^{-1}\frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

Example:

We saw in a previous example that the vectors $\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} 5 & 6 & 7 & 8 \end{bmatrix}$ are not parallel. So what angle do they form? Simple:

$$\theta = \cos^{-1}\frac{\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \bullet \begin{bmatrix} 5 & 6 & 7 & 8 \end{bmatrix}}{\sqrt{(1+4+9+16)(25+36+49+64)}} =$$

$$= \cos^{-1}\frac{70}{\sqrt{5130}} \approx 0.21$$

Of course this is in radians, corresponding to 60.32°. Or does it? Do degrees make any sense here?

Exactly: this may be cute, but what's the point? Even though the formula can be worked out, what do we do with an angle that has no meaning?

Excellent point, but it turns out that even though there is no traditional interpretation for this "*angle*", there are many applications that rely on it. Just have a little faith and you'll see.

Rather, are you sure that the formula of *Definition* 2.3.10 can always be computed? In mathematical jargon, is this formula *well defined*?

Why? What can go wrong?

When can we compute the inverse cosine included in the formula?

When the value of the fraction is a number from -1 to 1.

Oh: can that ratio generate a value bigger than 1?

It turns out that it can't, and I will grant you the joy of finding out why in the *Checkpoint* questions.

So you were correct: the formula is well defined, but in general be careful not to jump to conclusions, or assumptions. In math, and especially in linear algebra, that can be dangerous.

But since this definition is fine, let us take it a step further:

Definition

Two n-dimensional vectors **u** and **v** are said to be ***orthogonal***, or ***perpendicular***, if $\mathbf{u} \bullet \mathbf{v} = 0$, so that the angle between them is $\pi/2$.

Example:

The vectors $\mathbf{u} = \begin{bmatrix} 5 & 4 & -6 & 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 & -2 & 2 & 3 \end{bmatrix}$ are orthogonal, since:

$$\mathbf{u} \bullet \mathbf{v} = \begin{bmatrix} 5 & 4 & -6 & 5 \end{bmatrix} \bullet \begin{bmatrix} 1 & -2 & 2 & 3 \end{bmatrix} = 5 - 8 - 12 + 15 = 0$$

Example:

Any two different standard unit vectors are orthogonal, since they have non-zero components in different positions, hence their dot product is always 0.

This definition, silly as it may seem now, will turn out to be particularly useful later, so don't laugh at it. Just to show you the power of these extensions, look what good old friend shows up by considering them:

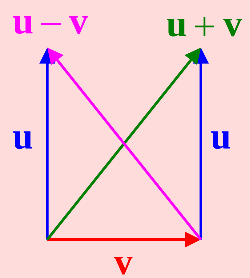
Technical fact:

The Pythagorean Theorem

If **u** and **v** are two orthogonal vectors in $\mathbb{R}^n$, then

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \quad \text{and} \quad \|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

Proof



First of all let me explain why this is a generalization of the Pythagorean theorem. If $n < 4$, so that we are dealing with geometric vectors, then the vectors **u-v** and **u+v** both represent the hypotenuse of a triangle whose other two sides are **u** and **v**, as the picture shows. Hence both equations express the fact that the square of the hippopotamus …well, you know the rest.

Now for the technical, algebraic part. By using the properties of the dot product we can do the following, starting from the left side of the left equation:

$$\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \bullet (\mathbf{u} - \mathbf{v}) = \mathbf{u} \bullet \mathbf{u} - 2\mathbf{u} \bullet \mathbf{v} + \mathbf{v} \bullet \mathbf{v}$$
$$= \mathbf{u} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

which proves that the left equation is correct. Do you see which property I used at each step? Make sure you do, since I am leaving the proof of the second equation to your exercises and, guess what: it uses exactly the same properties!

Example:

The vectors $\mathbf{u}=\begin{bmatrix}1 & 2 & -3 & 4\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}2 & 3 & 4 & 1\end{bmatrix}$ are orthogonal (check it!). In fact:

$$\|\mathbf{u}-\mathbf{v}\|^2 = \left\|\begin{bmatrix}-1 & -1 & -7 & 3\end{bmatrix}\right\|^2 = 1+1+49+9=60$$

$$\|\mathbf{u}+\mathbf{v}\|^2 = \left\|\begin{bmatrix}3 & 5 & 1 & 5\end{bmatrix}\right\|^2 = 9+25+1+25=60$$

And, sure enough:

$$\|\mathbf{u}\|^2+\|\mathbf{v}\|^2 = \left\|\begin{bmatrix}1 & 2 & -3 & 4\end{bmatrix}\right\|^2 + \left\|\begin{bmatrix}2 & 3 & 4 & 1\end{bmatrix}\right\|^2$$
$$=(1+4+9+16)+(4+9+16+1)=60$$

And even the triangle inequality holds, even though there are no geometrical triangles involved:

Technical fact

The triangle inequality

If $\mathbf{u}$ and $\mathbf{v}$ are any two vectors in Euclidean n-space, then:

$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\|+\|\mathbf{v}\|$$

Proof

Since we cannot rely on the geometry of the situation, we must use a purely algebraic argument.

To show that $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\|+\|\mathbf{v}\|$, it is sufficient to show that

$$\|\mathbf{u}+\mathbf{v}\|^2 \le \left(\|\mathbf{u}\|+\|\mathbf{v}\|\right)^2$$

(why?), so that is what we shall do. By using the definition of magnitude and the properties of the dot product we have that:

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\bullet(\mathbf{u}+\mathbf{v}) =$$
$$= \mathbf{u}\bullet\mathbf{u}+2\mathbf{u}\bullet\mathbf{v}+\mathbf{v}\bullet\mathbf{v} = \|\mathbf{u}\|^2+\|\mathbf{v}\|^2+2\mathbf{u}\bullet\mathbf{v}$$

By using the definition of angle in terms of dot product we see that:

$$\|\mathbf{u}\|^2+\|\mathbf{v}\|^2+2\mathbf{u}\bullet\mathbf{v} = \|\mathbf{u}\|^2+\|\mathbf{v}\|^2+2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$
$$\le \|\mathbf{u}\|^2+\|\mathbf{v}\|^2+2\|\mathbf{u}\|\|\mathbf{v}\|$$

The last step stems from the fact that the cosine function is never bigger than 1. But the last expression is:

$$\|\mathbf{u}\|^2+\|\mathbf{v}\|^2+2\|\mathbf{u}\|\|\mathbf{v}\| = \left(\|\mathbf{u}\|+\|\mathbf{v}\|\right)^2$$

The claim is therefore true!

Example:

The vectors $\mathbf{u}=\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}$ and $\mathbf{z}=\begin{bmatrix}2 & 3 & 4 & 5\end{bmatrix}$ are not orthogonal (why?) and we have:

$$\|\mathbf{u}+\mathbf{z}\| = \left\|\begin{bmatrix}3 & 5 & 7 & 9\end{bmatrix}\right\| = \sqrt{164} \approx 12.8$$

$$\|\mathbf{u}\|+\|\mathbf{z}\| = \sqrt{30}+\sqrt{54} \approx 12.83$$

Barely, but it works! What does that say about the angle between these two vectors?

And since we are at it, let me finish this first set of generalized definitions with another idea that will turn out to be more useful than you may think now.

Definition

The ***orthogonal projection*** (or simply ***projection***) of a Euclidian vector **u** on another vector **v** of the same dimension is given by:

$$proj_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \bullet \mathbf{v}}{\mathbf{v} \bullet \mathbf{v}}\mathbf{v}$$

In the same way, the ***component*** of **u** orthogonal to **v** is given by:

$$\mathrm{perp}_{\mathbf{v}}\mathbf{u} = \mathbf{u} - \mathrm{proj}_{\mathbf{v}}\mathbf{u}$$

Example:

The projection of $\mathbf{v} = \begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}$ on $\mathbf{z} = \begin{bmatrix}2 & 3 & 4 & 5\end{bmatrix}$ is:

$$\mathrm{proj}_{\mathbf{z}}\mathbf{v} = \frac{\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix} \bullet \begin{bmatrix}2 & 3 & 4 & 5\end{bmatrix}}{\begin{bmatrix}2 & 3 & 4 & 5\end{bmatrix} \bullet \begin{bmatrix}2 & 3 & 4 & 5\end{bmatrix}}\begin{bmatrix}2 & 3 & 4 & 5\end{bmatrix} = \frac{20}{27}\begin{bmatrix}2 & 3 & 4 & 5\end{bmatrix}$$

Of course this projection is parallel to **z**, as it should be. Also:

$$\mathrm{perp}_{\mathbf{z}}\mathbf{v} = \begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix} - \frac{20}{27}\begin{bmatrix}2 & 3 & 4 & 5\end{bmatrix} = \frac{1}{27}\begin{bmatrix}-13 & -6 & 1 & 8\end{bmatrix}$$

You may want to check that this last vector is in fact orthogonal to **v**.

Summary

- All geometric properties of geometric vectors can be extended to Euclidean vectors.
- However, these geometrical properties are such mostly in terminology, since the actual geometry of it is impossible to visualize.
- However, they do have useful applications, not only in theoretical mathematics, but also in applied sciences.
- The Pythagorean Theorem has a far reaching impact, even in higher dimensions!

Common errors to avoid

- Use the geometrical interpretation of these properties as convenient analogies, but do not rely on their intrinsic meaning.

Learning questions for Section LA 2-3

Review questions:

1. Explain the role of geometric concepts when dealing with Euclidean vectors of higher dimensions.

Memory questions:

1. What are the two other technical names used for the "length" of a Euclidean vector?

2. Which formula defines the "*angle*" between two vectors in higher dimensions?

Computation questions:

1. Determine the magnitude and the direction of the vector **v** whose tail is at the point $\mathbf{P}(3, 1, 0, -1, 2)$ and whose tip is at the point $\mathbf{Q}(1, -2, 4, -2, 0)$.

2. Determine the magnitude and the direction – as a unit vector – of the vector whose tail is at the point $(3, 1, 0, -1, 2)$ and whose tip is at $(1, -2, 4, -2, 0)$.

3. Determine whether projecting a 4-dimensional vector **v** on $\mathbf{u} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$ provides the same result as projecting **v** on $\mathbf{e}_1$ and $\mathbf{e}_2$ and then adding the two projections.

4. Given the vectors $\mathbf{u} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 & -1 & 1 & -2 \end{bmatrix}$, write **u** as the sum of a vector parallel to **v** and one orthogonal to it.

5. Given the vectors $\mathbf{u} = \begin{bmatrix} -3 & 1 & 2 & 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 4 & 0 & -8 & 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 6 & -1 & -4 & 3 \end{bmatrix}$, determine:
   a) the components of the vector $\mathbf{q} = \text{proj}_{\mathbf{v}}\mathbf{u}$
   b) whether the angle between the vectors **v** and **w** is acute or obtuse.

6. Determine the magnitude and the direction of the vector component of $\mathbf{u} = \begin{bmatrix} 1 & 1 & -1 & 2 \end{bmatrix}$ that is orthogonal to $\mathbf{v} = \begin{bmatrix} -2 & 1 & 3 & 0 \end{bmatrix}$.

7. Find the projection **p** of $\mathbf{u} = \begin{bmatrix} 1 & -2 & 3 & 1 \end{bmatrix}$ on $\mathbf{v} = \begin{bmatrix} 2 & 6 & -1 & 1 \end{bmatrix}$ and determine the angle between **v** and $\mathbf{u} - \mathbf{p}$.

8. Determine the norm and direction of the vector $\mathbf{v} = \begin{bmatrix} -2 & 3 & -1 & 4 & 2 \end{bmatrix}$ and present a vector orthogonal to it.

Theory questions:

1. Is the dot product commutative in $\mathbb{R}^n$?

2. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, is the projection of $\mathbf{u}$ on $\mathbf{v}$ parallel to the projection of $\mathbf{v}$ on $\mathbf{u}$?

3. What is the directional relation between a vector $\mathbf{v}$ and $\text{proj}_{\mathbf{v}}\mathbf{u}$?

4. Which vector operation is directly linked to the definition of vector magnitude?

5. What is the relation between the vectors $\mathbf{q} = \text{proj}_{\mathbf{v}}\mathbf{u}$ and $\mathbf{q}' = \text{proj}_{2\mathbf{v}}\mathbf{u}$?

6. What is the relation between the vectors $\mathbf{q} = \text{proj}_{\mathbf{v}}\mathbf{u}$ and $\mathbf{q}' = \text{proj}_{\mathbf{v}}2\mathbf{u}$?

7. What is meant as the angle between two Euclidean vectors?

Proof questions:

1. Prove that the magnitude of any Euclidean vector is positive.

2. Prove that the dot product of Euclidean vectors is
   a) commutative,
   b) associative,
   c) distributive with respect to vector addition, and
   d) positive definite.

3. Determine whether any of the following statements is true in an *n*-dimensional Euclidean space:
   a) Scalar multiplication is distributive with respect to the dot product.
   b) Dot product is associative.

4. Prove that $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$ for any Euclidean vector $\mathbf{v}$ and any scalar c.

5. Prove that for any non-zero vector $\mathbf{v}$, the vector $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector.

6. Prove that given any vector $\mathbf{v} = [v_1 \;\; v_2 \ldots v_n]$ whose first component is not zero, the vector $\mathbf{w} = [-v_2 \;\; v_1 \; 0 \; \ldots 0]$ is perpendicular to v. Then use this observation to devise a general strategy to construct a non-zero vector $\mathbf{w}$ perpendicular to a given non-zero vector $\mathbf{v}$.

7. Prove that the Pythagorean relations $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ and $\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ hold for any Euclidean vectors $\mathbf{u}$ and $\mathbf{v}$ that are orthogonal.

8. Prove that for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ the quantities $(\mathbf{u}+\mathbf{v})\bullet(\mathbf{u}-\mathbf{v})$ and $\|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$ are equal.

9. Determine under what conditions:
   a) the sum $x+y$ and the difference $x-y$ of two scalars have the same magnitude
   b) the sum $\mathbf{u}+\mathbf{v}$ and the difference $\mathbf{u}-\mathbf{v}$ of two vectors have the same magnitude

Application questions:

1. Consider the vector consisting of the marks obtained by a student in the five first-term engineering courses. Let us say that Pat and Chris are two such students, with vectors:

$$\mathbf{Pat} = [72 \quad 75 \quad 88 \quad 92 \quad 79] \qquad \mathbf{Chris} = [90 \quad 85 \quad 66 \quad 75 \quad 78]$$

   Compute the angle between Pat's vector on Chris' and the projection of the first on the second. Then discuss possible practical interpretations of these two procedures and their outcome in this application.

Templated questions:

1. Pick a dimension n and a whole number k between 1 and n and write down the components of $\mathbf{e}_k$.

2. Pick a dimension n and a vector $\mathbf{v}$ in $\mathbb{R}^n$ and compute $\|\mathbf{v}\|$.

3. Pick a dimension n and a vector $\mathbf{v}$ in $\mathbb{R}^n$ and compute $\mathbf{e}_{\mathbf{v}}$.

4. Pick a dimension n and two vectors in $\mathbb{R}^n$ and compute the angle between them.

5. Pick a dimension n and two vectors in $\mathbb{R}^n$ and compute the projection of the first on the second.

What questions do you have for your instructor?