Chapter 3: Linear systems and matrices *Section 1*

Linear equations and linear systems

What you need to know already:	*What you can learn here:*
➤ Basic concepts and properties of vectors. ➤ Basic concepts and properties of equations.	➤ The technical terminology relevant to linear equations and systems.

You are probably familiar with linear equations as they are studied in regular algebra, but let us start with the formal definition needed in the context of linear algebra, both to have it on record and to bring out the connection between the old and the new.

Definition

A ***linear equation in*** $\mathbb{R}^n$, or ***in n variables***, is an equation of the form $\mathbf{a} \bullet \mathbf{x} = c$, where:

- $\mathbf{a} = [a_1 \ a_2 \ \dots \ a_n]$ is an *n*-dimensional vector of ***coefficients***,
- $\mathbf{x} = [x_1 \ x_2 \ \dots \ x_n]$ is an *n*-dimensional vector of ***variables***,
- c is a constant ***scalar***.

Technical fact

By using the definition of dot product, any linear equation can be written as:

$$\mathbf{a} \bullet \mathbf{x} = c \quad \Leftrightarrow \quad a_1x_1 + a_2x_2 + \dots + a_nx_n = c$$

So, we define it by what it is, rather than what it isn't!

Yes, no need to make a long list of what it cannot contain. Just what it can. And it turns out that this approach has many more advantages.

Example:

I am sure that you could solve the equation $3x - 5 = 0$ even at the end of a very long and tiring day. But what I want you to notice here is that this equation may be written as $3x = 5$ (that would be your first step in the solution, right?) and therefore that this is a linear equation in one variable.

Example:

I am also sure that you will recognize the equation $y = 3x - 5$ as that of a line in the usual plane. But again notice that this equation may be written as

$$3x - y = 5 \quad \Leftrightarrow \quad \begin{bmatrix} 3 & -1 \end{bmatrix} \bullet \begin{bmatrix} x & y \end{bmatrix} = 5$$

Therefore, this is a linear equation in two variables.

As you can see, a linear equation is just a very simple equation in several variables, where each variable is multiplied only by a scalar, and not any other variable, and is not being squared, cubed, rooted, inverted or subject to any other kind of torturous manipulation! However, the vector notation allows us to use a definition that is very simple and does not require a long list of things we don't want.

That is true. Does it also simplify the way to solve such an equation?

It certainly does, but mostly when *n* is larger. But before explaining why that is so, let me explain what we mean by solving an equation. The required concepts are also likely very familiar to you, although at times they generate some confusion that can be eliminated by using the linear algebra approach.

Definition

A vector **v** is a ***solution*** of the linear equation $\mathbf{a} \bullet \mathbf{x} = c$ if $\mathbf{a} \bullet \mathbf{v} = c$, that is, if its components make the equation true.

If **v** is a solution of the linear equation $\mathbf{a} \bullet \mathbf{x} = c$, we say that the vector ***satisfies*** the equation.

The ***solution set*** of a linear equation consists of ***ALL*** its solutions.

To ***solve*** a linear equation means to find its solution set.

Example:

The equation $3x - 2y = 4$ is a linear equation with $\begin{bmatrix} 3 & -2 \end{bmatrix}$ as vector of coefficients and 4 as the constant. The vectors $\begin{bmatrix} 0 & -2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \end{bmatrix}$ are both solutions of the equation, since they both make the equation true (check it!).

Can we get *ALL* solutions of this equation? Of course we can, by using the standard algebraic method of isolating one variable:

$$3x - 2y = 4 \quad \Rightarrow \quad 3x - 4 = 2y \quad \Rightarrow \quad y = \frac{3}{4}x - 2$$

This means that the solution set consists of all vectors $\begin{bmatrix} a & b \end{bmatrix}$ that are of the form $\begin{bmatrix} a & \frac{3}{4}a - 2 \end{bmatrix}$.

Example:

The equation $2x - y - z = 0$ is a linear equation in 3 variables, with $\begin{bmatrix} 2 & -1 & -1 \end{bmatrix}$ as vector of coefficients and 0 as the constant.

To get all solutions of this equation we notice that we can write it as $z = 2x - y$, so that we can pick any values we want for *x* and *y* and then compute the corresponding needed value of *z*. So, for instance, $\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$ is a solution and so is $\begin{bmatrix} -3 & 2 & -8 \end{bmatrix}$. More generally, every solution will be of the form $\begin{bmatrix} a & b & 2a - b \end{bmatrix}$ for any values of *a* and *b*.

Here I am getting the feeling that you are using a lot of words to describe a simple concept.

You are correct, in the sense that linear equations are simple objects and that the terminology and notation I have presented so far seems somewhat of an overkill. But can you guess why we use it?

Because it can be generalized?

Exactly! And in the more general situation it does become useful and effective. One generalization that we consider immediately is to go from linear equations to linear systems.

Systems for doing what?

Misleading word, eh? A linear system is not a system or method to do something, but a very interesting mathematical problem with lots of applications. Here it is formally:

Definition

A ***linear system*** is a set of m linear equations in $\mathbb{R}^n$ that we consider simultaneously. For now, we shall denote such a system in one of these two equivalent ways:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = c_2 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = c_m \end{cases} \quad \text{or} \quad \begin{cases} \mathbf{a}_1 \bullet \mathbf{x} = c_1 \\ \mathbf{a}_2 \bullet \mathbf{x} = c_2 \\ \ldots \\ \mathbf{a}_m \bullet \mathbf{x} = c_m \end{cases}$$

A ***solution*** of a linear system is a vector in $\mathbb{R}^n$ that satisfies ***all*** of the equations of the system.

Example:

The linear system $\begin{cases} 2x - 3y = 4 \\ x + 3y = 6 \end{cases}$ consists of 2 equations in $\mathbb{R}^2$. The vector $\begin{bmatrix} 5 & 2 \end{bmatrix}$ satisfies the first equation, but not the second (check it!), therefore it is NOT a solution of the system. But $\begin{bmatrix} \frac{10}{3} & \frac{8}{9} \end{bmatrix}$ is: check it!

Example:

The system $\begin{cases} 2x - 3y + 2z = 4 \\ x + 4y - z = 6 \\ 3x + y + z = 10 \\ 3x - 10y + 5z = 2 \end{cases}$ consists of 4 equations in $\mathbb{R}^3$ and $\begin{bmatrix} 4 & 0 & -2 \end{bmatrix}$ is one of its solutions, since it makes all four equations true. How did I get it? How can we get the others? That's what we'll look at in the next sections.

Example:

The system $\begin{cases} 2x + 3y = 4 \\ 2x + 3y = 5 \end{cases}$ cannot possibly have any solutions, since any solution vector $\begin{bmatrix} a & b \end{bmatrix}$ would make the same expression, namely $2x + 3y$, equal to two different numbers.

So, not every system has a solution!

That is correct and, not surprisingly, there is some jargon related to that.

Definition

A linear system is said to be ***consistent*** if it has solutions, but it is said to be ***inconsistent*** if it has no solutions.

Example:

We saw in a previous example that the linear system $\begin{cases} 2x-3y=4 \\ x+3y=6 \end{cases}$ has the vector $\begin{bmatrix} \frac{10}{3} & \frac{8}{9} \end{bmatrix}$ as one solution, hence it is *consistent*.

On the other hand, we also saw that the system $\begin{cases} 2x+3y=4 \\ 2x+3y=5 \end{cases}$ cannot have any solutions, so it is *inconsistent*.

Now that I think back, I seem to remember that we studied how to solve a system in high school.

I am sure that you were presented *one* method to solve a *small* system and I am going to review it quickly in the next section. However, we shall soon meet a much faster and better method that, not surprisingly, is based on vectors. And in further preparation for that, here is a simple observation that we shall exploit extensively in our future work.

Technical fact

Every linear equation can be considered as a linear system consisting of just one equation.

Therefore, every fact we find about linear system and any method related to them also applies to linear equations.

That is reasonable and it sounds like a generalization!

Right, as it should be in the spirit of linear algebra! And, in the general spirit of mathematics, where we try to avoid unnecessary words and symbols, here is a convention that is standard within linear algebra.

Knot on your finger

As long as we are within the context of linear algebra and unless otherwise specified, whenever we refer to a ***system***, it will mean a ***linear system***.

But if we limit ourselves to linear systems, aren't we restricting our learning?

Yes, but linear systems will provide us with an amount of learning that is sufficient for one course! The topic of non-linear systems is a fascinating and rich one, that deserves more time and requires a basic knowledge of linear systems in order to be explored properly.

Moreover, in some cases we can change some simple non-linear systems to linear ones by using the familiar method of substitution. In that case, the methods we shall learn about linear systems will provides us with the means to solve non-linear ones.

Here is an example.

Example:

The system

$$\begin{cases} 2e^x - 3\sin y + 2\ln z = 4 \\ e^x + 4\sin y - \ln z = 6 \\ 3e^x + \sin y + \ln z = 10 \\ 3e^x - 10\sin y + 5\ln z = 2 \end{cases}$$

consists of 4 non-linear equations. However, if we make the substitutions:

$$u = e^x \quad ; \quad v = \sin y \quad ; \quad w = \ln z$$

the system becomes the linear system we say earlier, for which one of the solutions is $\begin{bmatrix}4 & 0 & -2\end{bmatrix}$. From this we know that one solution is obtained when:

$$4 = e^x \quad ; \quad 0 = \sin y \quad ; \quad -2 = \ln z$$

This tells us that any vector of the form $\begin{bmatrix}\ln 4 & k\pi & e^{-2}\end{bmatrix}$ is a solution.

And with that in mind, it is now time to start developing some methods for solving these systems. The *Learning questions* will help you clarify and confirm the terminology we have seen here.

Summary

- An equation is linear if it can be written as an equality between the dot product of a vector of coefficient and a vector of variables and a constant.
- A linear system is a simultaneous set of linear equations.
- A solution to a linear system is a vector that makes all equations of the system true.
- A consistent linear system is one that has solutions; an inconsistent system is one that does not have any.

Common errors to avoid

- Do not confuse the coefficients and the variables of a linear system. Often they are both represented by letters, but they play different roles.
- When checking if a vector is a solution of a linear system, you must check that it satisfies ALL its equations, not just some or even a majority!
- Even though in this course (and generally) by a system we shall usually mean a linear one, always confirm that this is the case.

Learning questions for Section LA 3-1: Linear equations and linear systems

Review questions:

1. Describe what identifies an equation as being linear.
2. Describe what identifies a system as being linear.
3. Explain what is meant by a system being consistent.

Memory questions:

1. What is the technical form of a linear equation?
2. What makes a system of equations linear?
3. Which adjective characterizes a linear system that has solutions?

Computation questions:

1. Verify that the vector [3 -7] is a solution of the linear equation $2x+5y=-29$.

2. Explain why the system $\begin{cases} 6e^x - y^3 + 2\sqrt{z} = 2 \\ e^x + y^3 - \sqrt{z} = 7 \end{cases}$ is not linear and then use suitable substitutions to change it into a linear system.

Determine whether each of the equations provided in questions 3-5 is linear in *x*, *y* and *z* or not and briefly explain why.

3. $3x+e^2 y - z\cos 1 = \ln 2$

4. $\cos x + 2y - 3z = 2$

5. $x^2 + y^2 - z^2 = k^2$

Determine, for each of the systems provided in questions 6-8, if it is linear or not. For each linear system explain what makes it linear and for each system that is not linear identify ALL the reasons why it is not.

6. $\begin{cases} 3x^2 + 2y^2 - z^2 = 0 \\ \pi xy - eyz + zx = 1 \\ \sqrt{2x} - \sqrt{22y} + \sqrt{11z} = 3 \end{cases}$

7. $\begin{cases} 3x + 2y - z = 0 \\ \pi x - ey + z = 1 \\ \sqrt{2}x - \sqrt{22}y + \sqrt{11}z = 3 \end{cases}$

8. $\begin{cases} 3^x + 2^y - z = 0 \\ x^{\pi} - y^e + z^x = 1 \\ \sqrt{2}x - \sqrt{22}y + \sqrt{11}z - 3 = 0 \end{cases}$

Theory questions:

1. Is it possible to construct a linear system of 7 equations in 2 variables?

2. If **v** is a solution to a linear system of 3 equations in 4 variables, what is the dimension of **v**?

3. Is the equation $\pi x + \sqrt{2}y - e^2 z = \ln 5$ linear?

4. Can some non-linear systems be changed to linear systems through suitable substitutions?

5. In a linear system, the vector of constants is expressed as a linear combination of what?

Application questions:

1. An internet media outlet has obtained $30 million in revenue for the sale or rent of the video of a certain movie. The movie could be rented for $6 or purchased for $15 and it was acquired by *v* viewers. Which linear system would allow us to figure out how many customers purchased the movie and how many rented it?

2. Sometimes non-linear systems can be solved by using a related linear system. For instance, if $2^x = 8^{y+1}$ and $9^y = 3^{x-9}$, what linear system allows us to find the values of *x* and *y*?

Templated questions:

1. Construct a linear system and explain why it is such.

2. Construct a system that is not linear and explain why it is such.

What questions do you have for your instructor?