Chapter 3: Linear systems and matrices | **Section 3**

Equivalent systems and elementary operations

What you need to know already:	What you can learn here:
➤ What it means to solve a system.	➤ Some key facts that will be used to develop a general method for solving linear systems.

The method of elimination and back-substitution is, in a way, a *brute-force* method, in that we systematically get one variable in terms of others, push it inside other equations and keep clearing the road in the same way until we get the solution. Although it works, this method has three major problems:

- it can lead to fairly complicated computational steps, especially when done by hand;
- it provides no insight into the nature of the system and its solutions;
- it does not offer any new mathematical tools.

In this section you will see a different approach that addresses all three problems and hence leads to a much better method for solving linear systems, a method that will be the focus of the next section.

The key idea of this approach can be seen clearly by considering the question: How would you rank these three systems from the easiest to the most difficult to solve?

$$\begin{cases} 2x+3y+z=25 \\ -x-2y+4z=-25 \\ 3x-y+2z=-2 \end{cases} \qquad \begin{cases} x \;\; +2y \;\; -4z \;\; =25 \\ \quad\; -y \;\; +9z \;\; =-25 \\ \qquad\quad -7z \;\; =14 \end{cases} \qquad \begin{cases} x=3 \\ y=7 \\ z=-2 \end{cases}$$

Well, the last one shows the solutions already, so there is nothing to be done, the middle one can be solved fairly easily by back substitution only and so the first one is the most difficult.

Very good. And yet all three systems have exactly the same solution! So, let us begin with some mandatory jargon:

Definition

Two systems and are said to be ***equivalent*** if they have exactly the same solution set.

And why is this jargon word useful?

Because I can now describe the new approach in a simple and short way.

Strategy for developing a more efficient method to solve linear systems

We can try to solve a system by ***changing*** it into simpler and simpler equivalent systems until the solution set is explicitly visible.

That looks like a good idea, but isn't it what elimination and back-substitution does?

It is, but it is only ONE way to implement the strategy. So, what we shall consider now is whether there are other ways to change a system to a simpler, equivalent one.

I bet that there are tons of ways!

The beauty of the method you are about to see is that it relies on only three very simple kinds of operations. First, make their acquaintance:

Definition

An ***elementary operation*** is a change we make to a linear system by following one of these three rules:

- ***Switch*** the position of two equations.
- ***Multiply*** one equation by a non-zero scalar, or ***divide*** by one.
- ***Add*** to both sides of an equation the same multiple of the corresponding side of another equation, or ***subtract*** it.

Example:
$$\begin{cases} 2x+3y+z=25 \\ -x-2y+4z=-25 \\ 3x-y+2z=-2 \end{cases}$$

Let me show you how to implement each type of operation starting from this original system. We can *switch* the first two equations, thus changing from the original to

$$\begin{cases} -x-2y+4z=-25 \\ 2x+3y+z=25 \\ 3x-y+2z=-2 \end{cases}$$

We can *multiply* the second equation by 3, thus changing from the original to

$$\begin{cases} 2x+3y+z=25 \\ -3x-6y+12z=-75 \\ 3x-y+2z=-2 \end{cases}$$

We can add to the first equation twice the second equation. This is a more convoluted operation, so here are the steps involved:

$$\begin{cases} 2x+3y+z=25 \\ -x-2y+4z=-25 \\ 3x-y+2z=-2 \end{cases}$$

$$\Rightarrow \begin{cases} 2x+3y+z+2(-x-2y+4z)=25+2(-25) \\ -x-2y+4z=-25 \\ 3x-y+2z=-2 \end{cases}$$

$$\Rightarrow \begin{cases} -y+9z=-25 \\ -x-2y+4z=-25 \\ 3x-y+2z=-2 \end{cases}$$

You may want to think of other elementary operations and perform them on the same original system.

I think I understand how these elementary operations work, but I don't see how they make the system simpler.

That is because I just showed an example of each operation without explaining why I picked it and what its simplifying effect was. But before I answer your objection more fully, let me state one important fact about these operations.

Technical fact

Each elementary operation changes a given system to one equivalent to it.

Proof

We need to show that once we perform one such operation, any solution of the original system is a solution of the new system and any solution of the new system is a solution of the original one. This is a fairly simple exercise, so I will ask you to do it in the *Learning questions*!

Now let me show you how to use these elementary operations to solve the system.

Example:

Starting from the same original system, I will first switch the first two rows as we did before, thus getting

$$\begin{cases} 2x+3y+z=25 \\ -x-2y+4z=-25 \\ 3x-y+2z=-2 \end{cases} \Rightarrow \begin{cases} -x-2y+4z=-25 \\ 2x+3y+z=25 \\ 3x-y+2z=-2 \end{cases}$$

The advantage of this form is that the first equation has a simpler coefficient for the first variable. I can make it even simpler by multiplying the current first row by -1, thus obtaining:

$$\Rightarrow \begin{cases} x+2y-4z=25 \\ 2x+3y+z=25 \\ 3x-y+2z=-2 \end{cases}$$

I can now add to the second equation -2 times the first, thus obtaining

$$\begin{cases} x+2y-4z=25 \\ 2x+3y+z-2(x+2y-4z)=25-2(25) \\ 3x-y+2z=-2 \end{cases}$$

$$\Rightarrow \begin{cases} x+2y-4z=25 \\ -y+9z=-25 \\ 3x-y+2z=-2 \end{cases}$$

Notice that the second equation now has no x term, and this is a useful simplification! I can obtain the same effect on the third equation by adding to it -3 times the first equation:

$$\begin{cases} x+2y-4z=25 \\ -y+9z=-25 \\ 3x-y+2z-3(x+2y-4z)=-2-3(25) \end{cases}$$

$$\Rightarrow \begin{cases} x+2y-4z=25 \\ -y+9z=-25 \\ -7y+14z=-77 \end{cases}$$

Can you now see why simplifying the first coefficient of the first equation was useful?

If I now divide the third equation by -7, we simplify it quite a bit:

$$\Rightarrow \begin{cases} x+2y-4z=25 \\ -y+9z=-25 \\ y-2z=11 \end{cases}$$

Now I can add the second equation to the third:

$$\Rightarrow \begin{cases} x+2y-4z=25 \\ -y+9z=-25 \\ y-2z+(-y+9z)=11+(-25) \end{cases}$$

$$\Rightarrow \begin{cases} x+2y-4z=25 \\ -y+9z=-25 \\ 7z=-14 \end{cases}$$

Notice that this is a system equivalent to the original one and is the second one that I showed at the beginning of the section! Now I could do back-substitution to get the solution, but instead I will continue through elementary operations, to show you more of how they work.

I divide the last one by 7:

$$\Rightarrow \begin{cases} x+2y-4z=25 \\ -y+9z=-25 \\ z=-2 \end{cases}$$

Subtract from the second 9 times the third:

$$\Rightarrow \begin{cases} x+2y-4z=25 \\ -y+9z-9z=-25+18 \\ z=-2 \end{cases}$$

$$\Rightarrow \begin{cases} x+2y-4z=25 \\ -y \quad =-7 \\ z=-2 \end{cases}$$

And see if you can identify the operations I am doing at each of the final steps:

$$\Rightarrow \begin{cases} x+2y-4z=25 \\ y \quad =7 \\ z=-2 \end{cases}$$

$$\Rightarrow \begin{cases} x+2y=17 \\ y=7 \\ z=-2 \end{cases} \Rightarrow \begin{cases} x=3 \\ y=7 \\ z=-2 \end{cases}$$

Et voila'! We have our solution!

I can see that it works, but it still seems pretty long to me!

It does seem so for two reasons:

- In showing you how the process works, I repeated several stages of the system, thus making the whole process seem longer.
- I am using the traditional notation for systems to keep track of the changes and that is NOT a good notation.

It looks like a good notation to me: what's a better one?

Ah, that is the topic of the next section, but let me finish this one with a simple, but very important observation:

Knot on your finger

When changing a system to an equivalent one by performing an ***elementary operation***, we are actually performing those same operations separately on the ***coefficients*** of the variables involved.

So?

What this tells us is that in order to implement this method of elementary operations efficiently, we can just focus on the coefficients and ignore the variables, as long as we keep track of their position. And that is what we shall do in the next section.

Summary

- When solving a system, it may be convenient to change it to an equivalent one, that is, to a different one that has the same solutions, especially if the new one is simpler.
- We can be sure that a system is changed to an equivalent one if we change it by using one of the three elementary operations of switching two equations, multiplying an equation by a non-zero scalar, or adding to an equation a multiple of another equation.
- By using elementary operations, it is possible to change any system to one whose solutions are explicitly stated.

Common errors to avoid

- Elementary operations are very basic and usually simple, but you will need to perform many of them. It is therefore easy to make computational errors that may lead to the wrong solutions. Be careful in the simple steps!

Learning questions for Section LA 3-3

Review questions:

1. Explain what it means for two systems to be equivalent and why it is an important concept.
2. Explain what elementary operations are.
3. Describe why elementary operations are important when solving linear systems.

Memory questions:

1. When are two systems equivalent?
2. List the three types of elementary row operations, being careful to use correct wording.

Theory questions:

1. What is special about the three types of operations done on a system that are called *elementary*?

2. Why are there no *Computation questions* for this section, even though one warning is about the potential for computational errors?

3. Why do we use only elementary row operations to solve a system?

Proof questions:

1. Prove that for each elementary operation there is another elementary operation that is its inverse, in the function sense that it undoes what the original operation did.

Templated questions:

1. Construct a system involving no more than 4 equations and no more than 3 variables and apply one elementary operation of each type to it.

What questions do you have for your instructor?