Chapter 3: Linear systems and matrices — **Section 5**

Gauss-Jordan elimination

What you need to know already:	What you can learn here:
➤ How to represent a linear system through a matrix. ➤ How to use elementary operations to solve a system. ➤ How to perform elementary operations on the rows of a matrix.	➤ The most efficient and effective way to solve a linear system. ➤ A method to manipulate matrices that will lead to many useful and interesting consequences.

I may have heard of Gauss before, but who was Jordan: his wife?

Not even close. Carl Frederick Gauss was a great German mathematician, whose name is associated with practically every area of mathematics you will see. Wilhelm Jordan was a German engineer (yes, one of those) who specialized in geodesy (do you know what that is?). You'll see the part they played in this method after we start exploring how the method works.

Is that the method Gauss and Jordan used to eliminate each other? Just kidding! But is this the same elimination method we saw is Section 3.2?

No, and for two reasons: it works on the augmented matrix instead of the system itself and it eliminates variables through a different sequence of steps than the old method. And there are even additional advantages that you will appreciate in later chapters.

First, let us see what we are trying to do. In Section 3-3, we looked at these systems:

$$\begin{cases} 2x+3y+z=25 \\ -x-2y+4z=-25 \\ 3x-y+2z=-2 \end{cases} \qquad \begin{cases} x \;+2y\; -4z \;=25 \\ \quad -y\; +9z \;=-25 \\ \qquad\quad -7z\;=14 \end{cases} \qquad \begin{cases} x=3 \\ y=7 \\ z=-2 \end{cases}$$

Notice that their augmented matrices are as follows:

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 25 \\ -1 & -2 & 4 & -25 \\ 3 & -1 & 2 & -2 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 2 & -4 & 25 \\ 0 & -1 & 9 & -25 \\ 0 & 0 & -7 & 14 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

We observed there that the second system is easier to solve than the first and the third is easier than the second. Looking at the systems there we noticed that this is because the equations involved were progressively simpler; looking at the matrices we can see that they have progressively more 0's and 1's.

This being linear algebra, and these being two particularly nice types of matrices, we need new words.

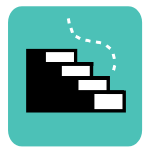
Definitions

A matrix is said to be in ***row echelon form (REF)*** if:

- Every entry below the first non-zero entry of each row is zero.
- No zero row is above a non-zero row.

The first non-zero entry (from the left) of a non-zero row of an *REF* matrix is called the ***leading coefficient***, or ***leading entry***, of that row.

Eche… what?



The French word *"echelon"* can refer to the rung of a ladder, but also to the steps of a hierarchical organization. In this case it is used as a combination of both meanings, since the pattern of zeros in a row echelon form reminds us of the steps of a (possibly irregular) ladder or staircase, but also because it can refer also to the lower degree of complexity of the system corresponding to such a matrix.

Example: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 5 & 6 \end{bmatrix}$

This matrix is in *REF* $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 5 & 6 \end{bmatrix}$, as you can see here, where it is written with the staircase highlighted and the leading coefficients in red. So are the following matrices:

$$\begin{bmatrix} -1 & 2 & 5 & -2 & 0 \\ 0 & 0 & 3 & 4 & 6 \\ 0 & 0 & 0 & 2 & -1 \end{bmatrix} \quad \begin{bmatrix} 7 & 1 & 2 & 4 & -1 \\ 0 & 1 & 5 & 3 & 2 \\ 0 & 0 & 0 & 6 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & -1 & 2 & 3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

On the other hand, the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -1 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & -1 & 7 \end{bmatrix}$ is not in *REF*, even though its zeros form a staircase pattern, since the first non-zero entry of the third row is under the first non-zero entry of the second, while the definition requires only zeros under any leading coefficient.

Another way to see this is that the steps of the staircase can only go done one row at a time, even though they may cover more than one column.

So, if this is the form we look for, how do we get it?

This is where we use the fact that *ERO*'s produce the same effects of row operations, hence keep the solutions of the system unchanged. And this is where the *Gaussian* part of the process comes in, since Gauss proposed it as an efficient method to solve systems.

Strategy to obtain an REF through Gaussian elimination

In order to change an augmented matrix into an equivalent *REF*:

1: If necessary, use a ***switch ERO*** to move a row whose first entry is not zero to the top position of the matrix.

2: If any of the rows below the first have a non-zero entry in the first column, use suitable ***multiply and add ERO***'s to change such entries to 0.

3: Keeping the first row and the leading 0's of the remaining rows as they are, ***repeat*** the first two steps on the smaller matrix consisting of the remaining rows without the leading 0's.

4: ***Repeat again*** the procedure until the *REF* is achieved.

To keep the procedure simple when working by hand on a matrix that consists of integers:

- pick the desired *ERO*'s so as to ***avoid creating fractions*** (unless you like working with fractions ☺), even if it means working bigger numbers;
- use suitable *ERO*'s to ***change an existing leading coefficient to 1***, if this simplifies later steps and does not create fractions.

Time out! This sounds awfully complicated!

Actually, this is a formal description of what we did in an earlier example, where it worked fine. It turns out that it works every time! But you do have a point in that this is a case of *easier done than said*, so let me give you a few pointers that may clarify it and then I will show you some examples.

Knots on your finger

When using Gaussian elimination, it is useful to:

- generate zeros in order, from the top down and from the left to the right.
- generate small integer numbers in the leading entry positions. The number 1 is best, but not necessary, especially if obtaining one means creating fractions.
- always use an upper row to change a lower row by using an *add ERO*.
- If all the entries of a row have a common factor, eliminate it by using a *multiply ERO*, so that all numbers become smaller.

Example:

$$\begin{cases} 2x+3y+z=25 \\ -x-2y+4z=-25 \\ 3x-y+2z=-2 \end{cases}$$

To solve this system, we start from its augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 25 \\ -1 & -2 & 4 & -25 \\ 3 & -1 & 2 & -2 \end{array}\right]$$

We can then switch the first two rows and multiply the new row by -1, so as to obtain a 1 in the leading position of the first row:

$$\overset{\mathbf{R}_1 \leftrightarrow \mathbf{R}_2}{\Rightarrow} \left[\begin{array}{ccc|c} -1 & -2 & 4 & -25 \\ 2 & 3 & 1 & 25 \\ 3 & -1 & 2 & -2 \end{array}\right] \overset{(-1)\mathbf{R}_1}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 25 \\ 2 & 3 & 1 & 25 \\ 3 & -1 & 2 & -2 \end{array}\right]$$

Next, we perform *add ERO*'s in order to obtain zeros under that first leading entry:

$$\overset{\mathbf{R}_2 - 2\mathbf{R}_1}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 25 \\ 0 & -1 & 9 & -25 \\ 3 & -1 & 2 & -2 \end{array}\right] \overset{\mathbf{R}_3 - 3\mathbf{R}_1}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 25 \\ 0 & -1 & 9 & -25 \\ 0 & -7 & 14 & -77 \end{array}\right]$$

We can now divide the last row by 7 to obtain smaller numbers and then do one more *add ERO* to get the *REF*:

$$\overset{\mathbf{R}_3 / (-7)}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 25 \\ 0 & -1 & 9 & -25 \\ 0 & 1 & -2 & 11 \end{array}\right] \overset{\mathbf{R}_3 + \mathbf{R}_2}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 25 \\ 0 & -1 & 9 & -25 \\ 0 & 0 & 7 & -14 \end{array}\right]$$

Example:
$$\begin{cases} 2x - y + 2z = 5 \\ x + 3y - z = 2 \\ 4x + 4y + z = -2 \end{cases}$$

Again, first we write the augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 5 \\ 1 & 3 & -1 & 2 \\ 4 & 4 & 1 & -2 \end{array}\right]$$

Now we could eliminate the 1 at the beginning of the second row by performing $\mathbf{R}_2 - \frac{1}{2}\mathbf{R}_1$, but this would create fractions all over. Instead we observe that the first entry in the second row is 1, which is an ideal leading entry, so we move it that row the top:

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 5 \\ 1 & 3 & -1 & 2 \\ 4 & 4 & 1 & -2 \end{array}\right] \overset{\mathbf{R}_1 \leftrightarrow \mathbf{R}_2}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 2 & -1 & 2 & 5 \\ 4 & 4 & 1 & -2 \end{array}\right]$$

Next we generate 0's in the rest of the first column:

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 2 & -1 & 2 & 5 \\ 4 & 4 & 1 & -2 \end{array}\right] \overset{\mathbf{R}_2 - 2\mathbf{R}_1}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & -7 & 4 & 1 \\ 4 & 4 & 1 & -2 \end{array}\right]$$

$$\overset{\mathbf{R}_3 - 4\mathbf{R}_1}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & -7 & 4 & 1 \\ 0 & -8 & 5 & -10 \end{array}\right]$$

At this point we have no row with a common multiple, but the two leading entries in R_2 and R_3 differ by one, so we play it smart to obtain another leading 1 and then another 0:

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & -7 & 4 & 1 \\ 0 & -8 & 5 & -10 \end{array}\right] \overset{\mathbf{R}_2 - \mathbf{R}_3}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & 1 & -1 & 11 \\ 0 & -8 & 5 & -10 \end{array}\right]$$

$$\overset{\mathbf{R}_3 + 8\mathbf{R}_2}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & 1 & -1 & 11 \\ 0 & 0 & -3 & 78 \end{array}\right]$$

Notice how I sometimes used *ERO*'s to obtain 0's, some other times I used them to make things simpler in the following steps, all the while staying away from fractions.

We could add a finishing touch by dividing $\mathbf{R}_3$ by -3 so as to get smaller numbers, but why bother?

Since we already have an *REF*, we can stop here and proceed to find the solution of the system by back substitution.

Before you look at how I work out the next example, try doing it yourself. Not only it is a good first exercise, but it will probably reveal an interesting technical fact that I will discuss next.

Example:

$$\begin{cases} 3x_1 - x_2 + 2x_3 - 2x_4 = 0 \\ 2x_1 + 2x_2 - x_3 + x_4 = 1 \\ 2x_1 - x_2 - 2x_3 - x_4 = 2 \\ x_1 + 3x_2 - 2x_3 + 4x_4 = 2 \end{cases}$$

The idea is to apply Gaussian elimination to this system, so go ahead and try…

Done? OK, so here are the steps I would use, but without rationale or explanation: see if you can supply those.

$$\left[\begin{array}{cccc|c} 3 & -1 & 2 & -2 & 0 \\ 2 & 2 & -1 & 1 & 1 \\ 2 & -1 & -2 & -1 & 2 \\ 1 & 3 & -2 & 4 & 2 \end{array}\right] \overset{\mathbf{R}_1 \leftrightarrow \mathbf{R}_4}{\Rightarrow} \left[\begin{array}{cccc|c} 1 & 3 & -2 & 4 & 2 \\ 2 & 2 & -1 & 1 & 1 \\ 2 & -1 & -2 & -1 & 2 \\ 3 & -1 & 2 & -2 & 0 \end{array}\right]$$

$$\begin{array}{c} \mathbf{R}_2 - 2\mathbf{R}_1 \\ \mathbf{R}_3 - 2\mathbf{R}_1 \\ \mathbf{R}_4 - 3\mathbf{R}_1 \\ \Rightarrow \end{array} \left[\begin{array}{cccc|c} 1 & 3 & -2 & 4 & 2 \\ 0 & -4 & 3 & -7 & -3 \\ 0 & -7 & 2 & -9 & -2 \\ 0 & -10 & 8 & -14 & -6 \end{array}\right] \overset{\mathbf{R}_3 - 2\mathbf{R}_2}{\Rightarrow} \left[\begin{array}{cccc|c} 1 & 3 & -2 & 4 & 2 \\ 0 & -4 & 3 & -7 & -3 \\ 0 & 1 & -4 & 5 & 4 \\ 0 & -10 & 8 & -14 & -6 \end{array}\right]$$

$$\overset{\mathbf{R}_2 \leftrightarrow \mathbf{R}_3}{\Rightarrow} \left[\begin{array}{cccc|c} 1 & 3 & -2 & 4 & 2 \\ 0 & 1 & -4 & 5 & 4 \\ 0 & -4 & 3 & -7 & -3 \\ 0 & -10 & 8 & -14 & -6 \end{array}\right] \begin{array}{c} \mathbf{R}_3 + 4\mathbf{R}_2 \\ \mathbf{R}_4 + 10\mathbf{R}_2 \\ \Rightarrow \end{array} \left[\begin{array}{cccc|c} 1 & 3 & -2 & 4 & 2 \\ 0 & 1 & -4 & 5 & 4 \\ 0 & 0 & -13 & 13 & 13 \\ 0 & 0 & -32 & 36 & 34 \end{array}\right]$$

$$\overset{-\frac{1}{13}\mathbf{R}_3}{\Rightarrow} \left[\begin{array}{cccc|c} 1 & 3 & -2 & 4 & 2 \\ 0 & 1 & -4 & 5 & 4 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -32 & 36 & 34 \end{array}\right] \overset{\mathbf{R}_4 + 32\mathbf{R}_3}{\Rightarrow} \left[\begin{array}{cccc|c} 1 & 3 & -2 & 4 & 2 \\ 0 & 1 & -4 & 5 & 4 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 4 & 2 \end{array}\right]$$

Did you get the same *REF*?

Probably not, and if you did, maybe one of your class mates didn't. And if by sheer luck we all got the same matrix, I can change mine from the one above by using another *ERO* and get a different *REF*, such as:

$$\overset{\mathbf{R}_1 + \mathbf{R}_2}{\Rightarrow} \left[\begin{array}{cccc|c} 1 & 4 & -6 & 9 & 4 \\ 0 & 1 & -4 & 5 & 4 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 4 & 2 \end{array}\right]$$

There! The point I am trying to make is the following:

Technical fact

The *REF* of a matrix is not unique, since different sequences of *ERO*'s may lead to different row-echelon forms, all corresponding to equivalent systems.

Although the *REF* is not unique, it does provide the solutions of the system, even though not explicitly. In fact, it provides lots more information on top of the solution, as we shall see later.

I also notice that in the last example you performed several ERO's at the same step. Is that allowed?

Technical fact

Several *ERO*'s may be performed at the same time, ***provided*** that none of them would affect the result of any of the others.

In particular, several *add ERO*'s may be performed at the same time when they are all adding ***multiples of the same row*** to the other rows.

This little innocent trick may add to the efficiency of the procedure, but make sure to use it wisely and not to mix up operations that interfere with each other!

But we are still half way through, since we know that there is an even easier form that we would like to get.

The one with 0' above and the explicit solution!

Exactly! And 1's as leading coefficients. Let's give this form a name:

Definition

A matrix is said to be in ***reduced row echelon form*** (***RREF***) if:

- It is in ***REF*** form.
- Every ***leading coefficient*** is 1.
- All entries ***above and below*** each leading coefficient are zero.

I am not sure of why we need the leading entries to be all 1.

Because our goal now is to see the solution explicitly and having 1 as leading entry means that the corresponding equation has the variable by itself, as needed for an explicit solution. It will become clear through the examples, but first let us see how to do it. The strategy is similar to Gaussian elimination to obtain an *RREF*, only in reverse order.

Strategy to obtain the RREF through Gauss-Jordan elimination

In order to change an *REF* matrix to its *RREF*:

1: ***Ignore*** any zero rows at the bottom of the matrix.

2: Use a ***multiply*** *ERO* to obtain 1 as leading entry in the last non-zero row.

3: If the entry above the leading coefficient of the last non-zero row is not 0, use suitable ***add*** *ERO*'s to change it to 0.

4: ***Repeat*** steps 2 and 3 on progressively higher rows and on all entries above leading coefficients until the *RREF* is achieved.

Again, use suitable *ERO*'s to keep the numbers small and integer if possible, although at this stage fractions may become unavoidable.

So Jordan just extended the process so that we would get the simplest form.

Well, some explorers open the path; others make it smoother and more comfortable! Let's see how this works in our examples.

Example: $\begin{cases} 2x+3y+z=25 \\ -x-2y+4z=-25 \\ 3x-y+2z=-2 \end{cases}$

If we complete the Gauss-Jordan elimination strategy on this system, we start from the *REF* we found earlier and follow the recommended steps:

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 25 \\ 0 & -1 & 9 & -25 \\ 0 & 0 & 7 & -14 \end{array}\right] \overset{\frac{1}{7}\mathbf{R}_3}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 25 \\ 0 & -1 & 9 & -25 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\overset{\substack{\mathbf{R}_1+4\mathbf{R}_3 \\ \mathbf{R}_2-9\mathbf{R}_3}}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 17 \\ 0 & -1 & 0 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right] \overset{\mathbf{R}_1+2\mathbf{R}_2}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & -1 & 0 & -7 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\overset{-\mathbf{R}_2}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Et voila: we have the *RREF* and the explicit solution, which can be written as the vector $\begin{bmatrix} 3 & 7 & -2 \end{bmatrix}$.

Example: $\begin{cases} 2x-y+2z=5 \\ x+3y-z=2 \\ 4x+4y+z=-2 \end{cases}$

To solve this system, we continue from the *REF* that we found earlier:

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & 1 & -1 & 11 \\ 0 & 0 & -3 & 78 \end{array}\right] \overset{-\frac{1}{3}\mathbf{R}_3}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & 1 & -1 & 11 \\ 0 & 0 & 1 & -26 \end{array}\right]$$

$$\overset{\substack{\mathbf{R}_1+\mathbf{R}_3 \\ \mathbf{R}_2+\mathbf{R}_3}}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 3 & 0 & -24 \\ 0 & 1 & 0 & -15 \\ 0 & 0 & 1 & -26 \end{array}\right] \overset{\mathbf{R}_1-3\mathbf{R}_2}{\Rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 21 \\ 0 & 1 & 0 & -15 \\ 0 & 0 & 1 & -26 \end{array}\right]$$

Therefore, the solution is $\begin{cases} x=21 \\ y=-15 \\ z=-26 \end{cases}$, or $\begin{bmatrix} 21 & -15 & -26 \end{bmatrix}$. Strange numbers, but obtained rather easily.

Example: $\begin{cases} 3x_1-x_2+2x_3-2x_4=0 \\ 2x_1+2x_2-x_3+x_4=1 \\ 2x_1-x_2-2x_3-x_4=2 \\ x_1+3x_2-2x_3+4x_4=2 \end{cases}$

And again, we start from the *REF* we have:

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & 4 & 2 \\ 0 & 1 & -4 & 5 & 4 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 4 & 2 \end{array}\right] \overset{\frac{1}{2}R_4}{\Rightarrow} \left[\begin{array}{cccc|c} 1 & 3 & -2 & 4 & 2 \\ 0 & 1 & -4 & 5 & 4 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 2 & 1 \end{array}\right]$$

$$\overset{\substack{R_1-2R_4 \\ R_2-2.5R_4 \\ R_3+0.5R_4}}{\Rightarrow} \left[\begin{array}{cccc|c} 1 & 3 & -2 & 0 & 0 \\ 0 & 1 & -4 & 0 & 1.5 \\ 0 & 0 & 1 & 0 & -0.5 \\ 0 & 0 & 0 & 2 & 1 \end{array}\right]$$

As you can see, in this case we cannot stay away from fractions any more, but at this point there is only a little way left to go and we can even use decimals to continue avoiding fractions!

$$R_1 + 2R_3 \quad R_2 + 4R_3 \Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -0.5 \\ 0 & 0 & 1 & 0 & -0.5 \\ 0 & 0 & 0 & 2 & 1 \end{array}\right] \quad R_1 - 3R_2 \quad 0.5R_4 \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0 & -0.5 \\ 0 & 0 & 1 & 0 & -0.5 \\ 0 & 0 & 0 & 1 & 0.5 \end{array}\right]$$

Tada! You state the solution. ☺

If you try any of the previous examples on your own, you will end up with exactly the same *RREF*, unless you make some computational mistake, no matter what sequence of ERO's you use. That is because:

Technical fact

The *RREF* of a matrix is unique, independently of the sequence of *ERO*'s used.

If two systems have the same *RREF* matrix, they have the same solution set.

Proof

Later I shall give you some more technical arguments to support this fact, but for now it should be enough to notice that by using *ERO*'s we have gone from system to equivalent system, hence keeping the same set of solutions. Since the *RREF* shows the solution set explicitly, and since we have not changed the position of the columns, which correspond to the variables, the *RREF* must show the same solutions, and hence will be the same, no matter what *ERO*'s we used.

And now for some good news for you, technology-loving types:

Knots on your finger

The Gauss-Jordan elimination strategy involves ***simple*** operations, but ***many*** of them, so that computers are ideal for its implementation.

Hence I recommend that you now pull out your calculator and find out how to enter a matrix and what command to use to obtain both *REF* and *RREF* from it.

Do I have to?

Yes, it's part of you practice work ☺

Let me try to stall: in the examples we always obtained a unique solution, but we have seen that systems sometimes may have no solution and sometimes infinitely many. What happens to the Gauss-Jordan process in that case?

It works even more efficiently! But you can't skip the practice work: I will answer your question in the next section.

Summary

- By using *ERO*'s in a well-managed way, we can solve any linear system through a fast, simple and programmable procedure.
- The idea of variable elimination leads to the *REF* of a matrix (and its system), while following it with back substitution leads to the *RREF*.

Common errors to avoid

- The Gauss-Jordan elimination procedure is like chess: it is easy to understand the individual moves, but it takes practice to combine them into a winning game, that is, a truly easy and fast method to obtain the solution of a system in every case.
- Watch out for the "*silly*" errors that can occur as a result of the large number of easy computations it requires.
- Remember that any matrix has infinitely many *REF*'s, but only one RREF. If you get a different *RREF*, someone has made a mistake, and it may be you!

Learning questions for Section LA 3-5

Review questions:

1. Describe how the Gauss-Jordan elimination method works.
2. Compare and contrast Gaussian elimination and Gauss-Jordan elimination.
3. Explain what the *REF* and the *RREF* of a matrix and of a system are.
4. Clarify the difference between the required steps of the Gauss-Jordan elimination procedure and the optional ones to use when performing it by hand.

Memory questions:

1. What form of a matrix is obtained as the result of Gaussian elimination?
2. What form of a matrix is obtained as the result of Gauss-Jordan elimination?
3. How many *REF*'s does a matrix have?
4. How many *RREF*'s does a matrix have?

Computation questions:

For each of the systems presented in questions 1-6, construct an *REF* and the *RREF* of the corresponding matrix and from them read off the solutions of the system. Write the solution set both in system and in vector form.

1. $\begin{cases} 3y-4z=1 \\ x+5y-3z=2 \\ 2z=4 \end{cases}$

2. $\begin{cases} 2x-5y+z=7 \\ 3y+2z=4 \\ 4x-4y+6z=1 \end{cases}$

3. $\begin{cases} 3x+y+z=4 \\ x-y+3z=5 \\ 2x+5y-z=1 \end{cases}$

4. $\begin{cases} 3x-y+z=4 \\ x+2y-3z=2 \\ 4x+y-2z=6 \end{cases}$

5. $\begin{cases} 7x_1+x_2+3x_3+2x_4=2 \\ x_1+x_2-2x_3-4x_4=1 \\ 3x_1+x_2-12x_3-16x_4=5 \end{cases}$

6. $\begin{cases} 3x+2y-z=-15 \\ 5x+3y+2z=0 \\ 3x+y+3z=11 \\ 6x-4y+2z=30 \end{cases}$

For each of the matrices presented in questions 7-12, construct a system for which it is the augmented matrix and then use Gauss-Jordan elimination to solve the system.

7. $\begin{bmatrix} 5 & 3 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 4 & 2 & 0 & 7 \end{bmatrix}$

8. $\begin{bmatrix} -4 & 2 & 0 \\ 3 & 7 & 1 \\ 8 & 8 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 2 & -1 & 3 & 5 & -1 \\ 0 & 0 & 3 & 1 & 2 \end{bmatrix}$

11. $\begin{bmatrix} 4 & 0 & -6 & -4 & -3 \\ 0 & 4 & -14 & 0 & 1 \\ 0 & 0 & 6 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 1 & -1 & 0 & 3 & 1 & 2 \\ 1 & 1 & 2 & 1 & -1 & 4 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{bmatrix}$

Solve the systems presented in questions 13-14 by using Gauss-Jordan elimination. The solution will be in terms of the generic constants *a*, *b* and *c*.

13. $\begin{cases} 2x+y+z=a \\ x-y-z=b \\ 3x+y+z=c \end{cases}$

14. $\begin{cases} x+y+z=a \\ 2x-y+2z=b \\ 2x+3y+3z=c \end{cases}$

The systems presented in questions 15-16 are consistent for only some value(s) of the constant *a*. Determine what such value(s) is and find the corresponding solutions.

15. $\begin{cases} 3x-2y \qquad =2 \\ -3x+6y+3z=1 \\ 3x+6y+6z=a \end{cases}$

16. $\begin{cases} x+y+z=a \\ 2x-y-3z=-2 \\ -x+2y+4z=a^2 \end{cases}$

17. The augmented matrix of a linear system is $\begin{bmatrix} a^2 & a & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 4 & -2 & 0 & 1 \end{bmatrix}$. Determine at least two values of *a* for which the system will be inconsistent.

Theory questions:

1. Is every *RREF* matrix also *REF*?

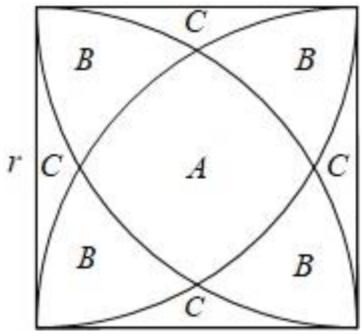
2. Are the leading coefficients of an *REF* required to equal 1?

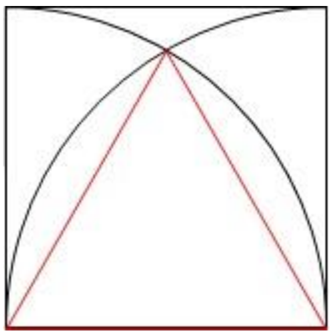
3. What feature of the *RREF* of an augmented matrix tells you that the system is inconsistent?

4. Why do we use only elementary row operations to solve a system?

Application questions:

1. In a square of side *r*, we draw four quarter circles as indicated in the picture. What are the areas of each of the three types of shapes so generated? Use geometric facts to set up a linear system of three equations in the variables *A*, *B* and C and then solve it by using Gauss-Jordan elimination. You may need to search for the formula of the area of an equilateral triangle, but I hope you know the ones for a square, a quarter circle and a sixth of a circle! They are all present in the second picture…





Templated questions:

1. Whenever using the Gauss-Jordan elimination method, identify where the Gaussian process ends and the "Jordan" phase begins.

What questions do you have for your instructor?