Chapter 3: Linear systems and matrices — **Section 6**

Homogeneous systems

What you need to know already:	What you can learn here:
➤ How to solve a linear system by using Gauss-Jordan elimination.	➤ Some special properties of a special type of systems that will gain more relevance as we delve deeper into the course.

There is a special type of systems that allows us some simplifications within the Gauss-Jordan elimination method. But more than that, it turns out that systems of this type are the cornerstone of all systems and play a pivotal role in many of the topics to come.

As the title of the section suggests, I am referring to *homogeneous* systems, and here is their definition.

Definition

A linear system is ***homogeneous*** if the constants of its equations are all 0:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{cases}$$

Therefore, a linear equation is ***homogeneous*** if its constant is 0:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$$

Example:

$$\begin{cases} 2x - 3y + 2z = 0 \\ x + 4y - z = 0 \\ 3x + y + z = 0 \\ 3x - 10y + 5z = 0 \end{cases}$$

This system is homogeneous and, therefore, each of its equations is also homogeneous.

The first nice fact about homogeneous system is rather obvious, but it will have important consequences.

Technical fact

Every ***homogeneous*** system is ***consistent***, since the **0** vector is a solution for it.

Since our goal when dealing with systems is to find *all* their solutions, when dealing with homogeneous system the focus is on finding the solutions other than the **0** vector. This leads to another common definition.

Definition

The **0** vector is called the ***trivial*** solution of a homogeneous system.

Knots on your finger

When dealing with a homogeneous system, the goal is to ***find its non-trivial solutions***.

To find such non-trivial solutions, we can use the Gauss-Jordan elimination method.

Example: $\begin{cases} 3x+5y-4z=0 \\ -3x-2y+4z=0 \\ 6x+y-8z=0 \end{cases}$

To solve this system, we change to the augmented matrix and perform suitable row operations

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \begin{matrix} R_2+R_1 \\ R_3-2R_1 \\ \Rightarrow \end{matrix} \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} R_3+3R_2 \\ \Rightarrow \end{matrix} \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2/3 \\ \Rightarrow \end{matrix} \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} R_1-5R_2 \\ \Rightarrow \end{matrix} \begin{bmatrix} 3 & 0 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1/3 \\ \Rightarrow \end{matrix} \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the *RREF* of the system, from which we can read the solutions:

$$\begin{cases} x-\frac{4}{3}z=0 \\ y=0 \end{cases} \Rightarrow \begin{cases} x=\frac{4}{3}z \\ y=0 \end{cases} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4z/3 \\ 0 \\ z \end{bmatrix}$$

But when using Gauss-Jordan, the last column never changed!

That is true and it will be true whenever we are dealing with a homogeneous system.

Knots on your finger

When applying the Gauss-Jordan elimination method to a ***homogeneous*** system, the ***column of constants can be ignored***, since it will consist entirely of zeros.

Row operations can instead be applied to the matrix of coefficients.

Warning bells

If the column of constants is omitted while applying Gauss-Jordan, care must be taken at the end to ***read the solution*** set properly.

Example:
$$\begin{cases} 2x-3y+2z=0 \\ x+4y-z=0 \\ 3x+y+z=0 \\ 3x-10y+5z=0 \end{cases}$$

To solve this system, we apply Gauss-Jordan to its matrix of coefficients:

$$\begin{bmatrix} 2 & -3 & 2 \\ 1 & 4 & -1 \\ 3 & 1 & 1 \\ 3 & -10 & 5 \end{bmatrix} \begin{matrix} R_2 \leftrightarrow R_1 \\ \Rightarrow \end{matrix} \begin{bmatrix} 1 & 4 & -1 \\ 2 & -3 & 2 \\ 3 & 1 & 1 \\ 3 & -10 & 5 \end{bmatrix}$$

$$\begin{matrix} R_2-2R_1 \\ R_3-3R_1 \\ R_4-3R_1 \\ \Rightarrow \end{matrix} \begin{bmatrix} 1 & 4 & -1 \\ 0 & -11 & 4 \\ 0 & -11 & 4 \\ 0 & -22 & 8 \end{bmatrix}$$

$$\begin{matrix} R_2-2R_1 \\ R_3-3R_1 \\ R_4-3R_1 \\ \Rightarrow \end{matrix} \begin{bmatrix} 1 & 4 & -1 \\ 0 & -11 & 4 \\ 0 & -11 & 4 \\ 0 & -22 & 8 \end{bmatrix} \begin{matrix} R_3-R_2 \\ R_4-2R_2 \\ \Rightarrow \end{matrix} \begin{bmatrix} 1 & 4 & -1 \\ 0 & -11 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} R_2+3R_1 \\ \Rightarrow \end{matrix} \begin{bmatrix} 1 & 4 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1-4R_2 \\ \Rightarrow \end{matrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We are now ready to read the solutions, but we need to remember the missing column:

$$\begin{cases} x-5z=0 \\ y+z=0 \end{cases} \Rightarrow \begin{cases} x=5z \\ y=-z \end{cases} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5z \\ -z \\ z \end{bmatrix}$$

I mentioned earlier that homogeneous systems are pivotal in the study of systems in general. The following concept will prove useful when we explore this fact later.

Definition

Given any linear system, its ***associated homogeneous system*** is the system obtained by changing all its constants to 0:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_2x_2 + \ldots + a_{2n}x_n = c_2 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = c_m \end{cases}$$

$$\Rightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_2x_2 + \ldots + a_{2n}x_n = 0 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{cases}$$

Technical fact

If **x** is a solution of the system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_2x_2 + \ldots + a_{2n}x_n = c_2 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = c_m \end{cases}$$

and **w** is a solution of the associated homogeneous system, then $\mathbf{x}+\mathbf{w}$ is also a solution of the original system.

Proof

In that case, for every row $\mathbf{r}_i$ of the matrix of coefficients we have that:

$$\mathbf{r}_i \bullet \mathbf{x} = c_i \quad ; \quad \mathbf{r}_i \bullet \mathbf{w} = 0$$

But this implies that:

$$\mathbf{r}_i \bullet (\mathbf{x}+\mathbf{w}) = \mathbf{r}_i \bullet \mathbf{x} + \mathbf{r}_i \bullet \mathbf{w} = c_i + 0 = c_i$$

In turn, this means that $\mathbf{x}+\mathbf{w}$ is a solution of the original system, as claimed.

Example:

$$\begin{cases} 2x - 3y + 2z = 4 \\ x + 4y - z = 6 \\ 3x + y + z = 10 \\ 3x - 10y + 5z = 2 \end{cases}$$

We saw in an earlier section, and you may want to check again, that $\mathbf{x} = \begin{bmatrix} 4 & 0 & -2 \end{bmatrix}$ is a solution of this system. We have also just seen that any vector of the form $\begin{bmatrix} 5z & -z & z \end{bmatrix}$ is a solution of its associated homogeneous system. That means that any vector of the form $\begin{bmatrix} 5z+4 & -z & z-2 \end{bmatrix}$ is also a solution of the system.

A natural question in linear algebra, one that is relevant to the last example, is whether there are other solutions to a system beyond the ones we have found, or whether this is all there is. You may want to find out by applying Gauss-Jordan to the original system. Later we shall find a general answer, with proof, to this question.

Summary

- Every homogeneous system has at least one solution, namely the zero vector.
- The main goal in solving a homogeneous system is to find its non-zero solutions.
- When applying the Gauss-Jordan method to a homogeneous system, there is no point in keeping the column of constants, so we use only the matrix of coefficients.
- The solution set of any linear system may be obtained by adding a single solution to the solution set of its associated homogeneous system.

Common errors to avoid

- When solving a homogeneous system, keep in mind that the column of constants is usually omitted while solving, but must be considered at the end, when writing down the solution set.

Learning questions for Section LA 3-6

Review questions:

1. Explain what a homogeneous system is.
2. Describe why homogeneous systems are very important when solving systems in general.
3. Identify which solutions are sought when solving a homogeneous system and why.

Memory questions:

1. What makes a system homogeneous?
2. Is there any inconsistent homogeneous system?
3. Which solution of a homogeneous system is called *trivial*?

Computation questions:

1. Write the homogeneous system whose matrix of coefficients is $\begin{bmatrix} 5 & 3 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 4 & 2 & 0 & 7 \end{bmatrix}$.
2. Write the augmented matrix of the homogeneous system associated to $\begin{cases} x-y+2z-w=-1 \\ 2x+y-2z-2w=-2 \\ -x-y-4z+w=2 \\ 4x-6z-4w=-3 \end{cases}$.

For each of questions 3-10, construct a homogeneous system whose matrix of coefficients is the given one and then use Gauss-Jordan elimination to solve the system.

3. $\begin{bmatrix} 5 & 3 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 4 & 2 & 0 & 7 \end{bmatrix}$

4. $\begin{bmatrix} -4 & 2 & 0 \\ 3 & 7 & 1 \\ 8 & 8 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 2 & -1 & 3 & 5 & -1 \\ 0 & 0 & 3 & 1 & 2 \end{bmatrix}$

7. $\begin{bmatrix} 4 & 0 & -6 & -4 & -3 \\ 0 & 4 & -14 & 0 & 1 \\ 0 & 0 & 6 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 1 & -1 & 0 & 3 & 1 & 2 \\ 1 & 1 & 2 & 1 & -1 & 4 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 1 & -1/2 & 3/2 & -1/2 & 5 \\ 0 & 1 & -2 & 5/2 & -11/2 \\ 0 & 0 & 1 & 1/2 & 5/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 & 0 & 0 & 3/7 \\ 0 & 1 & 0 & -8/5 & -2/35 \\ 0 & 0 & 1 & 6/5 & -11/35 \end{bmatrix}$

11. Obtain the solution set of the system $\begin{cases} 3x+2y-\ z=0 \\ \quad 3y+\ 4z=0 \\ -9y-12z=0 \end{cases}$ by performing suitable row operations on its matrix of coefficients

Proof questions:

1. Provide a very short proof of why both the system $\begin{cases} 3x-2y+7z=1 \\ 6x+y-2z=2 \\ 9x-y+5z=3 \end{cases}$ and its associated homogeneous system have infinitely many solutions.

Templated questions:

1. Construct a homogeneous system with at most 4 equations and at most 4 variables and solve it.

What questions do you have for your instructor?