Chapter 3: Linear systems and matrices — **Section 7**

Number of solutions of a system

What you need to know already:	What you can learn here:
➤ How to solve a linear system by using Gauss-Jordan elimination.	➤ How to detect the number of solutions of the system from an *REF*.

Don't we know already that a system can have either no solution, or one or infinitely many?

Yes, but we are now going to expand on that in two directions:

- by seeing how the augmented matrix of a system tells us the number of solutions and
- by seeing that there are different versions of the "*infinitely many*" case that needs to be analyzed.

You may remember that the *REF* of a matrix is not unique, but it turns out that all possible *REF*'s of the augmented matrix of a system have some special features related to its number of solutions.

We start with a very easy situation.

Technical fact

A linear system is inconsistent if its *REF* (any one of them) has a leading coefficient in the last column.

Proof

If this the case, the row to which that coefficient belongs will be of the form

$$\left[0 \quad 0 \quad \ldots \quad 0 \;\middle|\; k\right],\ k \neq 0$$

If we read this row as an equation, it tells that a non-zero number (on the right) is equal to 0 (on the left). This is clearly impossible, so that no matter what values we pick for the variables, we end up with a false equation. Hence the system has no solutions.

I refer you to the exercises to become practically familiar with this easy property of the *REF* of a system.

The next fact is equally easy to check and it leads directly to another useful property.

Technical fact

The process of going from an *REF* to its *RREF* does not change the position of the leading entries of the matrix.

Proof

Well, just look back at how the Gauss-Jordan process works and check that in going from *REF* to *RREF* no leading coefficient can become 0, thus making another entry a leading coefficient.

Technical fact

A linear system has a unique solution if its *REF* (any one of them) has one leading coefficient in each of the variable columns.

Proof

If there is one leading coefficient for each variable column of the *REF*, there will also be one for each variable column in the *RREF*. But in that case the rows of the *RREF* all correspond to equations of the form:

$$x_i = k_i$$

and therefore there is only one possible solution for the system.

This leaves us with only one more option to consider.

Technical fact

A consistent linear system has infinitely many solutions if the number of leading coefficients in its *REF* (any one of them) is less than the number of variables.

Proof

In this case each non-zero row of the *RREF* will correspond to an equation that includes one variable that has a leading coefficient and some that don't. For any choice of values for the variables without a leading coefficient (and there are infinitely many of them!) there is one possible set of values for the variables with a leading coefficient that makes all equations, and hence the system, satisfied. Therefore, the system has infinitely many solutions.

Slow down! I can't see what is going on anymore!

Good point, so let me show you an example.

Example: $\begin{bmatrix} 1 & 2 & -4 & 25 \\ 0 & -1 & 9 & -25 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Let's say that this is the *REF* of the augmented matrix of a system. We can see that the system has 3 variables, but only the first two have leading coefficients. This will not change by going to the *RREF*, which is as follows (check it!):

$$\begin{bmatrix} 1 & 0 & 14 & -25 \\ 0 & 1 & -9 & 25 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we can read that the original system is equivalent to either of these two:

$$\begin{cases} x + 14z = -25 \\ y - 9z = 25 \end{cases} \Rightarrow \begin{cases} x = -25 - 14z \\ y = 25 + 9z \end{cases}$$

But notice that for any value we pick for z the last version automatically gives us values for x and y. We have infinitely many choices for z, so the system has infinitely many solutions.

Notice that the thinking behind this last example does not change in case we have a different number of variables or of leading coefficients. It only requires that we have a consistent system and fewer leading coefficients than variables, just like the *Fact* claims.

Before I give you more examples to explore, please notice that the previous *Technical facts* exhaust all possibilities. Therefore, we can summarize them in the following way:

Knots on your finger

A linear system with m variables and whose *REF* (any one of them) has k leading coefficients will have:

- ***no solutions*** (inconsistent) if and only if one of those coefficients is in the column of constants.
- ***one*** (unique) solution if and only if $m=k$ and no leading coefficient is in the column of constants.
- ***infinitely many solutions if*** $m>k$ and no leading coefficient is in the column of constants.

Let's see how we can use all this.

Example:

$$\begin{cases} 3x+4y+z=1 \\ 2x+3y \quad\ \ =0 \\ 4x+3y-z=-2 \\ -7x \quad\ \ +7z=11 \end{cases}$$

If we solve this system by using Gaussian elimination, we start from its augmented matrix and end up with an *REF*, say:

$$\left[\begin{array}{ccc|c} 3 & 4 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \\ -7 & 0 & 7 & 11 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -7 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Did you get the same one? If not, did you get the same number of leading coefficients and in the same positions?

The fact that we have 3 variables, 3 leading coefficients and none in the last column tells us that this system has a unique solution. You may want to complete the Gauss-Jordan process to figure out what that solution is.

If we change the constant in the last equation to 12 we should end up with an inconsistent system, since the only solution we have now makes the left side of the last equation equal to 11, not 12. In fact, the *REF* of the new matrix becomes:

$$\left[\begin{array}{ccc|c} 3 & 4 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \\ -7 & 0 & 7 & 12 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -7 & -8 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

and the leading coefficient of the last row is in the column of constants, confirming that the system is not consistent.

Example:

$$\left[\begin{array}{cccc|c} 2 & 1 & 1 & -2 & 1 \\ 1 & 1 & -1 & -1 & -1 \\ 6 & 0 & 1 & -9 & -2 \\ 5 & -1 & 2 & -8 & 3 \end{array}\right]$$

If we try to solve the system whose augmented matrix is the one above, we get:

$$\mathbf{R}_1 \leftrightarrow \mathbf{R}_2 \begin{bmatrix} 1 & 1 & -1 & -1 & | & -1 \\ 2 & 1 & 1 & -2 & | & 1 \\ 6 & 0 & 1 & -9 & | & -2 \\ 5 & -1 & 2 & -8 & | & 3 \end{bmatrix} \begin{matrix} \mathbf{R}_2 - 2\mathbf{R}_1 \\ \mathbf{R}_3 - 6\mathbf{R}_1 \\ \mathbf{R}_4 - 5\mathbf{R}_1 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & -1 & | & -1 \\ 0 & -1 & 3 & 0 & | & 3 \\ 0 & -6 & 7 & -3 & | & 4 \\ 0 & -6 & 7 & -3 & | & 8 \end{bmatrix}$$

$$\mathbf{R}_4 - \mathbf{R}_3 \begin{bmatrix} 1 & 1 & -1 & -1 & | & -1 \\ 0 & -1 & 3 & 0 & | & 3 \\ 0 & -6 & 7 & -3 & | & 4 \\ 0 & 0 & 0 & 0 & | & 4 \end{bmatrix}$$

Although we are not at the *REF* stage yet, we can stop here, since the presence of a leading coefficient in the last column tells us that the system is inconsistent, both because of what the corresponding equation reads (0=4) and because there is no way to get the leading coefficient out of that column.

Example:

$$\begin{bmatrix} 1 & 2 & 1 & -3 & 0 & 1 & | & 1 \\ 1 & 2 & 2 & -3 & 0 & 3 & | & 2 \\ 2 & 4 & 3 & -6 & 1 & 5 & | & 5 \\ 5 & 10 & 8 & -15 & 2 & 13 & | & 12 \end{bmatrix}$$

Changing this augmented matrix (can you write down its system?) to *REF* form by hand is easy, but tedious. However your friendly calculator will tell you that its *RREF* is:

$$\begin{bmatrix} 1 & 2 & 0 & -3 & 0 & -1 & | & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & | & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & | & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

We can see that there are no leading coefficients in the last column, so the system is consistent. But it is a system with 6 variables and only 3 leading coefficients, therefore it will have infinitely many solutions.

You may want to write down the equations that provide such solutions.

Definition

Any variable of a consistent linear system whose corresponding column of the augmented matrix does NOT have a leading coefficient in a *REF* is called a ***free*** variable.

A variable that is not free, that is, a variable whose corresponding column of the augmented matrix has a leading coefficient in a *REF*, is called a ***basic*** variable.

Example:

$$\begin{cases} 3x + 4y + z = 1 \\ 2x + 3y \qquad = 0 \\ 4x + 3y - z = -2 \\ -7x \qquad + 7z = 11 \end{cases}$$

One *REF* corresponding to the above system is:

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & -2 & | & -2 \\ 0 & 0 & -7 & | & -8 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This is a consistent system, but each column has a leading coefficient, hence there are no free variables, which is consistent with the fact that the system has a unique solution: no variable can have a choice.

Example: $\left[\begin{array}{cccc|c} 2 & 1 & 1 & -2 & 1 \\ 1 & 1 & -1 & -1 & -1 \\ 6 & 0 & 1 & -9 & -2 \\ 5 & -1 & 2 & -8 & 3 \end{array}\right]$

You can check that one *REF* for this augmented matrix is:

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -1 \\ 0 & -1 & 3 & 0 & 3 \\ 0 & 0 & -11 & -3 & -14 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right]$$

This time there is no leading coefficient in the fourth column, but we cannot say that the fourth variable is free, because the system is inconsistent.

Example: $\left[\begin{array}{cccccc|c} 1 & 2 & 1 & -3 & 0 & 1 & 1 \\ 1 & 2 & 2 & -3 & 0 & 3 & 2 \\ 2 & 4 & 3 & -6 & 1 & 5 & 5 \\ 5 & 10 & 8 & -15 & 2 & 13 & 12 \end{array}\right]$

We have seen earlier that the RREF of this matrix is:

$$\left[\begin{array}{cccccc|c} 1 & 2 & 0 & -3 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This time the system is consistent and the second, fourth and sixth variables do not have a leading coefficient in their column, hence they are free.

If you write down the corresponding equations and the isolate the basic variables on the left side, you will see that the free variables are exactly the ones that are moved to the right side of the equations and that we can give them any value we choose. That is why we call them free.

Example: $\begin{cases} 7x+2y+z-3w=5 \\ x+2y+4z=1 \end{cases}$

Solving this system can be done easily by using Gauss-Jordan elimination:

$$\left[\begin{array}{cccc|c} 7 & 2 & 1 & -3 & 5 \\ 1 & 2 & 4 & 0 & 1 \end{array}\right] \mathbf{R}_1 \leftrightarrow \mathbf{R}_2 \left[\begin{array}{cccc|c} 1 & 2 & 4 & 0 & 1 \\ 7 & 2 & 1 & -3 & 5 \end{array}\right]$$

$$\mathbf{R}_2 - 7\mathbf{R}_1 \left[\begin{array}{cccc|c} 1 & 2 & 4 & 0 & 1 \\ 0 & -12 & -27 & -3 & -2 \end{array}\right]$$

This is only an REF, not an RREF, but it is sufficient for us to determine that the system has infinitely many solutions and that z and *w* are its free variables. Try to write down its solution set.

Be careful to not read too much into the uniqueness of the *RREF*!

Warning bells

While the number of free variables is a characteristic of the system, which variables are free depends on the order in which we write such variables within the system and may end up being different. Usually this is not an issue, but keep that in mind just in case ☺

This looks like an efficient way to get information.

It is! And it is related to other information that we shall uncover later.

Do I see more terminology coming?

You bet! This is linear algebra!

Definition

The ***rank*** of a matrix is the number of ***leading coefficients*** in any one of its *REF*, including its *RREF*.

This is also equivalent to the number of ***non-zero rows*** in any *REF* or *RREF*.

Notice that this definition really depends on matrices, not on systems and we shall see later what else it produces when we study matrices more in depth. But for now, let's start simple.

Technical fact

A linear system has no solutions, and is therefore ***inconsistent***, if and only if any *REF* of its augmented matrix has a leading entry in the augmented column.

This is the same as saying that it is ***inconsistent*** if and only if the ***rank of its augmented matrix is greater than the rank of the matrix of coefficients***.

Proof

If the *REF* of the augmented matrix has a leading entry in the last column, the row that includes that entry states that 0 (the left side, which only includes 0's) is equal to the non-zero leading entry. This is impossible and hence the system cannot have a solution.

On the other hand, if there are no leading entries in the last column of the *REF*, neither are there any in the *RREF*, since the process of going from *REF* to *RREF* does not change the position of any leading entry. Therefore, the *RREF* provides an explicit solution and hence such a solution exists.

In the same way you can check the following two statements:

Technical fact

A consistent linear system has a ***unique solution*** if and only if the rank of its matrix of coefficients is the same as the number of variables.

A consistent linear system has ***infinitely many solutions*** if and only if the rank of its matrix of coefficients is less than the number of variables.

Notice that if a system is consistent and has infinitely many solutions, we are also able to count how many free variables it has. This information will become useful later, so that, once again, it deserves a definition.

Definition

The number of free variables of a consistent system is called the ***dimension*** of its solution set.

So, the rank plus the dimension of the solution set gives the number of variables, right?

Yes, a fact that we shall solidify later in the course. Time to move to other ideas.

Summary

- The number of solutions of a linear system can be determined by comparing the number and position of the leading coefficients of any one of its *REF*'s.
- When a system has infinitely many solutions, it may be informative to determine how many free variables give rise to those solutions. Such number is called the dimension of the solution set.

Common errors to avoid

- There is no need to go all the way to the RREF to determine the number of solutions: any *REF* will do.

Learning questions for Section LA 3-7

Review questions:

1. For a given system, describe how the relationships among *n* (number of variables), r_A (rank of the augmented matrix) and r_C (rank of the coefficient matrix) determines the presence of no solution, one solution or infinitely many solutions respectively.

Memory questions:

1. What is the rank of a matrix?
2. What kind of systems can have 3 and only 3 solutions?
3. For what type of systems is the rank of the matrix of coefficients less than the rank of the augmented matrix?

Computation questions:

Each of questions 1-5 provides an *REF* of the augmented matrix of a linear system. Determine:

a) The rank of the augmented matrix
b) The rank of the matrix of coefficients
c) The number of solutions of the system
d) The number of free variables
e) The number of solutions of the associated homogeneous system

1. $\begin{bmatrix} 1 & 2 & -4 & 25 \\ 0 & -1 & 9 & -25 \\ 0 & 0 & 0 & 14 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & -4 & 25 \\ 0 & -1 & 9 & -25 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 4 & 0 & -6 & -4 & -3 \\ 0 & 4 & -14 & 0 & 1 \\ 0 & 0 & 6 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -1/2 & 3/2 & -1/2 & 5 \\ 0 & 1 & -2 & 5/2 & -11/2 \\ 0 & 0 & 1 & 1/2 & 5/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 7 & 1 & 3 & 2 & 2 \\ 0 & 6 & -17 & -30 & 5 \\ 0 & 0 & 5 & 6 & -11/7 \end{bmatrix}$

6. $\begin{bmatrix} 7 & 1 & 0 & 1 & 2 \\ 0 & 0 & -1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

7. Determine the condition on *a*, *b* and *c* for which the system $\begin{cases} 3x - y + 2z = a \\ x + 4y - 2z = b \\ 5x - 6y + 6z = c \end{cases}$ has solutions and, in such case, determine how many solutions it has.

8. Determine the value of *a* for which the system $\begin{cases} 3x + y + z = 3 \\ x - y + 3z = 5 \\ 4x + ay - 4z = -4 \end{cases}$ has infinitely many solutions and determine the solution set in vector form.

9. Use suitable substitutions to change the system $\begin{cases} 6e^x - y^3 + 2\sqrt{z} = 2 \\ e^x + y^3 - \sqrt{z} = 7 \end{cases}$ into a linear system and use it to explain why this system will have infinitely many solutions.

Theory questions:

1. How many solutions does a system have if the rank of the matrix of coefficients is less than the rank of the augmented matrix?

2. How many solutions can a system of 3 equations in four variables have?

3. How many solutions can a homogeneous system of 3 equations in four variables have?

4. Can the rank of the augmented matrix of a linear system be larger than the number of variables?

5. Is it possible for a non-linear system to have more than one solution, but not infinitely many?

6. If **A** is an $n \times n$ matrix of rank n, how many solutions does the system $\mathbf{Ax} = \mathbf{1}_n$ have?

Templated questions:

1. Construct a simple linear system, compute one of its *REF*'s and, from it, determine the number of solutions of the system.

What questions do you have for your instructor?