Chapter 4: Matrix Algebra *Section 2*

Special types of matrices

What you need to know already:	*What you can learn here:*
➤ What a matrix is. ➤ The basic terminology and notation used for matrices.	➤ The names of characteristics of some special types of matrices that will play a major role in later developments.

There are certain types of matrices whose special properties allow them to play an important role in what we'll do later. It is time for you to make their acquaintance.

Don't laugh if you feel that some of the following definitions are ridiculously simple: they will prove more useful and effective than what their definition seems to suggest. Because of that, some of the following examples may seem redundant, but I still hope they will clarify the concepts.

Definition

An ***n×n*** matrix is said to be a ***square matrix***, since its array of entries can be thought of as a square.

Example:

The matrix $\mathbf{A}=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ is a square matrix, while

$\mathbf{B}=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}$ and $\mathbf{A}=\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}$ are not.

Definition

An $m\times n$ matrix whose entries are all 0's is called the ***zero*** matrix and is denoted by $\mathbf{0}_{mn}$, or simply $\mathbf{0}$, if the dimensions are obvious.

Example:

The matrix $\mathbf{0}_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the zero matrix of $M_{2\times3}$.

Similarly, $\mathbf{0}_{2\,2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the zero matrix of $M_{2\times2}$ and it is a square matrix.

Definition

A square matrix whose elements outside the diagonal are all 0's is called a ***diagonal*** matrix.

Example:

The matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix.

The matrix $\mathbf{B} = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is square, but not diagonal, since it has non-zero entries off the diagonal.

Definition

An $n\times n$ diagonal matrix whose diagonal entries are all 1's is called the ***n-dimensional identity matrix*** and denoted by $\mathbf{I}_n$, or simply $\mathbf{I}$, if the dimension is obvious.

Example:

The matrix $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the 2×2 identity matrix, while the matrix

$\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the 3×3 identity matrix.

Definition

A square matrix whose elements below the diagonal are all 0's is called an ***upper triangular*** matrix.

A square matrix whose elements above the diagonal are all 0's is called a ***lower triangular*** matrix.

Example:

The matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is upper triangular. Notice that the presence of a 0 entry above the diagonal does not affect this. The deciding feature is that all entries *below* the diagonal are 0.

The matrix $\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & -3 & 0 & 0 \\ 5 & 0 & 2 & 0 \\ 3 & 1 & 1 & 2 \end{bmatrix}$ is lower triangular.

Is that it? This looks like a very simple section!

You are correct, but, as I said earlier, the definitions provided here will be important in what follows. So, learn them well and, since they are simple, no excuse if you use them incorrectly, eh?

Summary

- There are certain types of matrices that have special features making them important and useful.
- It is important to be familiar with these types of matrices and to recognize them when they occur.

Common errors to avoid

- Don't underestimate this section! It may be easy, but it is important!

Learning questions for Section LA 4-2

Review questions:

1. Describe what feature makes a matrix, respectively, square, diagonal, identity, upper triangular or lower triangular.

Memory questions:

1. When is a matrix square?
2. When is a matrix diagonal?
3. What is an identity matrix?
4. What is an upper triangular matrix?

Computation questions:

Determine the special types to which each of the matrices in question 1-5 belongs.

1. $\mathbf{A}=\begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 1 \\ -1 & 1 & -3 \end{bmatrix}$

2. $\mathbf{A}=\begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ -1 & 1 & -3 \end{bmatrix}$

3. $\mathbf{A}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

4. $\mathbf{A}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

5. $\mathbf{A}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Theory questions:

1. For what values of k and p can a $k \times p$ matrix be diagonal?

2. If a square matrix is in ref form, is it upper triangular, lower triangular or neither?

Proof questions:

1. Explain why every diagonal matrix is both upper and lower triangular.

Templated questions:

1. Construct a small ($m \le 4, n \le 4$) matrix and then determine of what type it is.

What questions do you have for your instructor?