Chapter 4: Matrix algebra — **Section 3**

Basic matrix operations

What you need to know already:	What you can learn here:
➤ What a matrix is. ➤ The basic special types of matrices	➤ How to combine and manipulate matrices in ways that will be fruitful later.

Since matrices can be viewed as vectors, there is a natural way of generalizing to them the basic operations we have used for vectors.

Technical fact

Two matrices of the same dimensions may be ***added*** together entry-by entry, just as it is done with two vectors:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & a_{ij} & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & b_{ij} & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \dots & \dots & a_{ij}+b_{ij} & \dots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}$$

Technical fact

A matrix can be ***multiplied by a scalar*** entry-by-entry, just as it is done for vectors:

$$c\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & a_{ij} & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} & \dots & ca_{1n} \\ ca_{21} & ca_{21} & \dots & ca_{2n} \\ \dots & \dots & ca_{ij} & \dots \\ ca_{m1} & ca_{m1} & \dots & ca_{m1} \end{bmatrix}$$

SO, I guess we can also construct linear combinations, right?

Of course, and the way to do it is the obvious one.

Technical fact

A ***linear combination*** of matrices can then be defined by using these two operations, just as it is done for vectors.

$$c\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & a_{ij} & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} + d\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & b_{ij} & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} =$$

$$\begin{bmatrix} ca_{11}+db_{11} & ca_{12}+db_{12} & \dots & ca_{1n}+db_{1n} \\ ca_{21}+db_{21} & ca_{22}+db_{22} & \dots & ca_{2n}+db_{2n} \\ \dots & \dots & ca_{ij}+db_{ij} & \dots \\ ca_{m1}+db_{m1} & ca_{m2}+db_{m2} & \dots & ca_{mn}+db_{mn} \end{bmatrix}$$

This is one more reason in support of my original choice of notation for vectors: matrices are universally denoted by using square brackets and they are a special type of vectors, just as vectors are a special type of matrices, so why not use the same notation? Well, I do, so probably the onus to justify their choice is on those who don't! ☺

And I thought that mathematics is free of controversies!

Not really! Controversies are an integral part of the mathematical world, since it is a world populated by humans. But for now, the more urgent task is to become more familiar with matrices, whatever notation we use, so let's get back there.

Yes, but these operations seem fairly simple.

They are and they never seem to cause problems to students, so I will give you here is just one basic example. If you need more practice, infinitely many numbers will provide infinitely many more examples and exercise!

Example:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 5 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -2 & 7 & 8 \\ 2 & 3 & -1 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$

We can add the first two matrices and obtain:

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 5 & 0 \end{bmatrix} + \begin{bmatrix} -2 & 7 & 8 \\ 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 9 & 7 \\ 4 & 8 & -1 \end{bmatrix}$$

We can also multiply **C** by a scalar:

$$\frac{1}{2}\mathbf{C} = \begin{bmatrix} 0.5 & 1 \\ 1.5 & -2 \end{bmatrix}$$

Or we can take a linear combination of the first two:

$$2\mathbf{A}-3\mathbf{B} = 2\begin{bmatrix} 3 & 2 & -1 \\ 2 & 5 & 0 \end{bmatrix} - 3\begin{bmatrix} -2 & 7 & 8 \\ 2 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 4 & -2 \\ 4 & 10 & 0 \end{bmatrix} + \begin{bmatrix} 6 & -21 & -24 \\ -6 & -9 & 3 \end{bmatrix} = \begin{bmatrix} 12 & -17 & -26 \\ -2 & 1 & 3 \end{bmatrix}$$

Notice, however, that in this example we cannot add **A** and **C** or **B** and **C**, since in each case the two matrices involved do not have the same dimensions.

Why can't we embed in into a larger matrix?

We certainly can, as long as we are aware that we are doing it. To refresh your memory on this concept, here is how it applies to matrices.

Definition

An $m \times n$ matrix is ***embedded*** in $M_{(m+h)\times(n+k)}$ by adding h rows and k columns all containing 0 entries. Unless otherwise needed, the additional rows and columns are added at the end.

Example: $\mathbf{A} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 5 & 0 \end{bmatrix}$

We can embed this matrix into $M_{2\times5}$ by adding two columns:

$$\mathbf{A} \quad \Rightarrow \quad \mathbf{A}_{2\times5} = \begin{bmatrix} 3 & 2 & -1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 0 \end{bmatrix}$$

Or we can embed it into $M_{4\times3}$ by adding two rows:

$$\mathbf{A} \quad \Rightarrow \quad \mathbf{A}_{4\times3} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Or we can embed it into $M_{3\times4}$ by adding one row and one column:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & -1 & 0 \\ 2 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Remember that these rows and columns can be added in different positions, but usually they are added at the end. Notice the connection of this embedding with something we have done earlier.

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Embedding an $m \times n$ matrix $\mathbf{A}$ into $M_{m\times(n+k)}$ is equivalent to augmenting it with the matrix $\mathbf{0}_{mk}$.

But earlier we augmented a matrix with a single column, while now we are adding several and we are even adding rows!

Yes, but this is not a problem. Rather it is one of the advantages of generalizing: you can do the same thing in more than one way. Notice how this embedding allows us to add matrices of different dimensions when we need to:

Example: $\mathbf{A} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 5 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$

We can embed $\mathbf{C}$ into $M_{2\times3}$, so that we can add it to $\mathbf{A}$:

$$\mathbf{A} + \mathbf{C}_{2\times3} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 5 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 3 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -1 \\ 5 & 1 & 0 \end{bmatrix}$$

Can we also truncate a matrix?

Of course, and this leads to another word for your linear algebra vocabulary.

Definition

A matrix **A** in $M_{(m+h)\times(n+k)}$ can be ***truncated*** to an $m\times n$ matrix by deleting h rows and k columns that are not wanted or needed.

In particular, the matrix obtained from a matrix **A** by truncating the p-th row and q-th column is called the ***(p, q) minor*** of **A** and it will be denoted by $\mathbf{A}_{(p,q)}$.

Example: $\mathbf{B}=\begin{bmatrix}2&3&1&0\\-2&1&5&2\\3&2&-2&1\\-1&0&7&3\end{bmatrix}$

We can truncate this matrix by deleting $\mathbf{r}_2$, $\mathbf{c}_1$ and $\mathbf{c}_4$:

$$\mathbf{B}=\begin{bmatrix}2&3&1&0\\-2&1&5&2\\3&2&-2&1\\-1&0&7&3\end{bmatrix}\Rightarrow\begin{bmatrix}3&1\\2&-2\\0&7\end{bmatrix}$$

Or we can see that its (2,3) minor is:

$$\mathbf{B}=\begin{bmatrix}2&3&1&0\\-2&1&5&2\\3&2&-2&1\\-1&0&7&3\end{bmatrix}\Rightarrow\mathbf{B}_{(2,3)}=\begin{bmatrix}2&3&0\\3&2&1\\-1&0&3\end{bmatrix}$$

I am confident that, except for the adding of more jargon, you will find all these operations very easy. And so is the last operation we'll look at here.

Since a matrix is a set of numbers organized over rows and columns, we can explore what happens by switching the roles of rows and columns. It turns out that this is a very useful operation: it will help us in many situations and will open new horizons of uses. But for now, let's just get used to the idea and the terminology.

Definition

Given a matrix **A**, its ***transpose*** is the matrix $\mathbf{A}^T$ obtained by writing all rows as columns in the same order.

$$\mathbf{A}=\begin{bmatrix}a_{11}&a_{12}&\dots&a_{1n}\\a_{21}&a_{22}&\dots&a_{2n}\\\dots&\dots&a_{ij}&\dots\\a_{m1}&a_{m2}&\dots&a_{mn}\end{bmatrix}\Leftrightarrow\mathbf{A}^T=\begin{bmatrix}a_{11}&a_{21}&\dots&a_{m1}\\a_{12}&a_{22}&\dots&a_{m2}\\\dots&\dots&a_{ji}&\dots\\a_{1n}&a_{2n}&\dots&a_{mn}\end{bmatrix}$$

Here is another way to look at it:

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The entry in position (i, j) of $\mathbf{A}^T$ is the entry in position (j, i) of **A**.

In particular, the elements a_{ii} of the diagonal are unchanged during transposition.

If **A** is an $m\times n$ matrix, then its transpose $\mathbf{A}^T$ is an $n\times m$ matrix.

Example:

The transpose of $\mathbf{B} = \begin{bmatrix} 0 & -2 & 1 \\ 3 & 1 & 4 \\ -2 & 2 & -1 \end{bmatrix}$ is $\mathbf{B}^T = \begin{bmatrix} 0 & 3 & -2 \\ -2 & 1 & 2 \\ 1 & 4 & -1 \end{bmatrix}$.

The transpose of $\mathbf{C} = \begin{bmatrix} -1 & 2 \\ 1 & 3 \\ 2 & 5 \end{bmatrix}$ is $\mathbf{C}^T = \begin{bmatrix} -1 & 1 & 2 \\ 2 & 3 & 5 \end{bmatrix}$.

Technical fact

For any matrix **A,** it is true that $\left(\mathbf{A}^T\right)^T = \mathbf{A}$

Well, duh! You are just reversing rows and columns again!

Yes, but this small and obvious observation will come in handy on several occasions, so it deserves to be highlighted. It also leads to another easy addition to your linear algebra vocabulary.

Definition

A matrix is ***symmetric*** if it is equal to its transpose, that is, if $\mathbf{A} = \mathbf{A}^T$.

Example: $\mathbf{B} = \begin{bmatrix} 0 & -2 & 1 \\ -2 & 1 & 4 \\ 1 & 4 & -1 \end{bmatrix}$

This matrix is symmetric, as you can tell from writing down its transpose. Can you see how the entries in mirror image positions across the diagonal are related?

The matrix $C = \begin{bmatrix} -1 & 2 \\ 1 & 3 \\ 2 & 5 \end{bmatrix}$ cannot possibly be symmetric since it has more rows than columns, so that when we reverse them we cannot possibly get back the same matrix.

The matrices of this last example highlight the following observations, which may be classified as technical facts, but are so simple that I chose not to. ☺

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All ***symmetric*** matrices are ***square*** matrices

The entries that are ***mirror image*** across the diagonal of a symmetric matrix are ***equal***: $a_{ij} = a_{ji}$.

On the other hand, the following property is very important and, though it looks obvious, it requires a little proof.

Technical fact

The ***sum*** of symmetric matrices is symmetric.

Proof

If two square matrices **A** and **B** have the same dimensions and are symmetric, any entry of their sum is given by $(\mathbf{A}+\mathbf{B})_{ij} = \mathbf{A}_{ij} + \mathbf{B}_{ij}$. But since they are symmetric we can write:

$$(\mathbf{A}+\mathbf{B})_{ij} = \mathbf{A}_{ij} + \mathbf{B}_{ij} = \mathbf{A}_{ji} + \mathbf{B}_{ji} = (\mathbf{A}+\mathbf{B})_{ji}$$

But this shows that each entry is equal to the one in its symmetric position, so that the whole matrix is symmetric

Why do we need a proof for such a simple fact?

Because some mathematical statements that seem very simple, indeed obvious, sometimes are false! Remember that we are not dealing with individual real numbers, but with sets of numbers endowed with a specific ordering organization, so that things can go wrong and will go wrong in situations that we shall soon see.

To give you more experience with this, the following fact is also easy to prove, and I leave it to you to develop its proof as an exercise. But we do need to prove it to be sure of its truth. We are getting deeper into the theoretical nature of linear algebra!

Technical fact

The transpose of the sum of two matrices of the same dimensions equals the sum of their transposes:

$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$

Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 4 & -5 & 0 \\ 3 & -1 & 5 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -3 & 5 & -2 \\ 5 & 1 & 0 \\ -2 & 0 & 7 \end{bmatrix}$

Let's verify this seemingly simple fact in this case.

You can check that only **B** is symmetric and that:

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} -2 & 7 & -1 \\ 9 & -4 & 0 \\ 1 & -1 & 12 \end{bmatrix} \Rightarrow (\mathbf{A}+\mathbf{B})^T = \begin{bmatrix} -2 & 9 & 1 \\ 7 & -4 & -1 \\ -1 & 0 & 12 \end{bmatrix}$$

However:

$$\mathbf{A}^T + \mathbf{B}^T = \begin{bmatrix} 1 & 4 & 3 \\ 2 & -5 & -1 \\ 1 & 0 & 5 \end{bmatrix} + \begin{bmatrix} -3 & 5 & -2 \\ 5 & 1 & 0 \\ -2 & 0 & 7 \end{bmatrix} = \begin{bmatrix} -2 & 9 & 1 \\ 7 & -4 & -1 \\ -1 & 0 & 12 \end{bmatrix}$$

And the two are the same.

Please notice that this exercise does NOT constitute a proof, since we only verified one particular case, not the general one. The latter is for you to do!

Summary

- Matrices can be added, multiplied by a scalar and combined into linear combinations, since they are vectors.
- The additional structure into rows and columns allows us to devise an additional operation on transposition: switching rows and columns.

Common errors to avoid

- Don't underestimate the concepts of this section: they are easy, but they are useful and used! So learn how to handle them properly.

Learning questions for Section LA 4-3

Review questions:

1. Describe how to perform a linear combination of matrices.
2. Explain how to transpose a matrix.

Memory questions:

1. What method is used to add two matrices or multiply a matrix by a scalar?
2. When is it possible to perform a linear combination of two matrices?
3. How is a matrix transposed?
4. What do we call a matrix that is equal to its transpose?
5. In general, what is a different but equivalent way to write $(\mathbf{A}+\mathbf{B})^T$?

Theory questions:

1. Can a 3×4 matrix be symmetric?
2. Are diagonal matrices symmetric?
3. What are the dimensions of the transpose of a 3×5 matrix?
4. What type of matrix is $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 1 \\ -1 & 1 & -3 \end{bmatrix}$?

Proof questions:

1. Prove that the sum of two symmetric matrices is symmetric.
2. Prove that if **A** is upper triangular, **B** is lower triangular, and they have the same dimensions, then **A**+**B** is neither upper nor lower triangular, unless both **A** and **B** are both diagonal.
3. Prove that the transpose of an upper triangular matrix is lower triangular and vice-versa.
4. Prove that the transpose of a linear combination of matrices is the same linear combination of the transposes.
5. Prove that a linear combination of symmetric matrices is symmetric.
6. Prove or disprove that for any matrix **A** the matrix $\mathbf{A} - \mathbf{A}^T$ is symmetric.

Templated questions:

1. Construct two matrices of the same dimensions (not too big and not too small!) and add them.
2. Construct a scalar and a matrix (not too big and not too small!) and multiply them.
3. Construct two scalars and two matrices of the same dimensions (not too big and not too small!) and construct a linear combination of them.
4. Construct a matrix (not too big and not too small!) and transpose it.

What questions do you have for your instructor?